# THE CAUCHY PROBLEM FOR THE TWO DIMENSIONAL EULER-POISSON SYSTEM

DONG LI AND YIFEI WU

Abstract. The Euler-Poisson system is a fundamental two-fluid model to describe the dynamics of the plasma consisting of compressible electrons and a uniform ion background. In the 3D case Guo [8] first constructed a global smooth irrotational solution by using the dispersive Klein-Gordon effect. It has been conjectured that same results should hold in the two-dimensional case. In our recent work [12], we proved the existence of a family of smooth solutions by constructing the wave operators for the 2D system. In this work we completely settle the 2D Cauchy problem.

## 1. Introduction

The Euler-Poisson system is one of the simplest two-fluid models used to describe the dynamics of a plasma consisting of moving electrons and ions. In this model the heavy ions are assumed to be immobile and uniformly distributed in space, providing only a background of positive charge. The light electrons are modeled as a charged compressible fluid moving against the ionic forces. Neglecting magnetic effects, the governing dynamics of the electron fluid is given by the following Euler-Poisson system in $(t,x) \in [0,\infty) \times \mathbb{R}^d$,

$$\begin{cases} \partial_t n + \nabla \cdot (n\mathbf{u}) = 0, \\ m_e n(\partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u}) + \nabla p(n) = en\nabla\phi, \\ \Delta\phi = 4\pi e(n-n_0). \end{cases} \tag{1.1}$$

Here $n = n(t,x)$ and $\mathbf{u} = \mathbf{u}(t,x)$ denote the density and average velocities of the electrons respectively. The symbol $e$ and $m_e$ denote the unit charge and mass of electrons. The pressure term $p(n)$ is assumed to obey the polytropic $\gamma$-law, i.e.

$$p(n) = An^\gamma, \tag{1.2}$$

where $A$ is the entropy constant and $\gamma \geq 1$ is called the adiabatic index. The term $en\nabla\phi = (-ne)\cdot(-\nabla\phi)$ quantifies the electric force acting on the electron fluid by the positive ion background. Note that the electrons carry negative charge $-ne$. We assume at the equilibrium the density of ions and electrons are both a constant denoted by $n_0$. To ensure charge neutrality it is natural to impose the condition

$$\int_{\mathbb{R}^d} (n-n_0)dx = 0.$$

The boundary condition for the electric potential $\phi$ is a decaying condition at infinity, i.e.

$$\lim_{|x|\to\infty} \phi(t,x) = 0. \tag{1.3}$$

The first and second equations in (1.1) represent mass conservation and momentum balance of the electron fluid respectively. The third equation in (1.1) is the usual Gauss law in electrostatics. It computes the electric potential self-consistently through the charge distribution $(n_0 e - ne)$. The Euler-Poisson system is one of the simplest two-fluid model in the sense that the ions are treated as uniformly distributed sources in space and they appear only as a constant $n_0$ in the Poisson equation. This is a very physical approximation since $m_{ion} \gg m_e$ and the heavy ions move much more slowly than the light electrons.

Throughout the rest of this paper, we shall consider an irrotational flow

$$\nabla \times \mathbf{u} = 0 \tag{1.4}$$

which is preserved in time. For flows with nonzero curl the magnetic field is no longer negligible and it is more physical to consider the full Euler-Maxwell system.

We are interested in constructing smooth global solution around the equilibrium $(n, \mathbf{u}) \equiv (n_0, 0)$. To do this we first transform the system (1.1) in terms of certain perturbed variables. For simplicity set all physical constants $e$, $m_e$, $4\pi$ and $A$ to be one. To simplify the presentation, we also set $\gamma = 3$ although other cases of $\gamma$ can be easily treated as well. Define the rescaled functions

$$\begin{aligned} u(t,x) &= \frac{n(t/c_0, x) - n_0}{n_0}, \\ \mathbf{v}(t,x) &= \frac{1}{c_0}\mathbf{u}(t/c_0, x), \\ \psi(t,x) &= 3\phi(t/c_0, x), \end{aligned}$$

where the sound speed is $c_0 = \sqrt{3}n_0$. For convenience we set $n_0 = 1/3$ so that the characteristic wave speed is unity. The Euler-poisson system (1.1) in new variables take the form

$$\begin{cases} \partial_t u + \nabla \cdot \mathbf{v} + \nabla \cdot (u\mathbf{v}) = 0, \\ \partial_t \mathbf{v} + \nabla u + \nabla \left(\frac{1}{2}u^2 + \frac{1}{2}|\mathbf{v}|^2\right) = \nabla \psi, \\ \Delta \psi = u. \end{cases} \tag{1.5}$$

Taking one more time derivative and using (1.4) then transforms (1.5) into the following quasi-linear Klein-Gordon system:

$$\begin{cases} (\Box + 1)u = \Delta \left(\frac{1}{2}u^2 + \frac{1}{2}|\mathbf{v}|^2\right) - \partial_t \nabla \cdot (u\mathbf{v}), \\ (\Box + 1)\mathbf{v} = -\partial_t \nabla \left(\frac{1}{2}u^2 + \frac{1}{2}|\mathbf{v}|^2\right) + (1 - \Delta^{-1})\nabla\nabla \cdot (u\mathbf{v}). \end{cases} \tag{1.6}$$

For the above system, in the 3D case Guo [8] first constructed a global smooth irrotational solution by using dispersive Klein-Gordon effect and adapting Shatah's normal form method. It has been conjectured that same results should hold in the two-dimensional case. In our recent work [12], we proved the existence of a family of smooth solutions by constructing the wave operators for the 2D system. The 2D problem with radial data was studied in [13]. In this work we completely settle the 2D Cauchy problem for general non-radial data. The approach we take in this paper is based on a new set-up of normal form transformation developed by Germain, Masmoudi and Shatah [3, 4, 5]. Roughly speaking (and over-simplifying quite a bit), the philosophy of the normal form method is that *one should integrate parts whenever you can in either (frequency) space or time.* The part where one cannot integrate by parts is called the set of space-time resonances which can often

be controlled by some finer analysis provided the set is not so large or satisfies some frequency separation properties. The implementation of such ideas is often challenging and depends heavily on the problem under study. In fact the heart of the whole analysis is to choose appropriate functional spaces utilizing the fine structure of the equations. The main obstructions in the 2D Euler-Poisson system are slow(non-integrable) $\langle t\rangle^{-1}$ dispersion, quasilinearity and nonlocality caused by the Riesz transform. Nevertheless we overcome all such difficulties in this paper. To put things into perspective, we review below some related literature as well as some technical developments on this problem.

The main difficulty in constructing time-global smooth solutions for the Euler-Poisson system comes from the fact that the Euler-Poisson system is a hyberbolic conservation law with zero dissipation for which no general theory is available. The "Euler"-part of the Euler-Poisson system is the well-known compressible Euler equations. Indeed in (1.1) if the electric field term $\nabla\phi$ is dropped, one recovers the usual Euler equations for compressible fluids. In [21], Sideris considered the 3D compressible Euler equation for a classical polytropic ideal gas with adiabatic index $\gamma > 1$. For a class of initial data which coincide with a constant state outside a ball, he proved that the lifespan of the corresponding $C^1$ solution must be finite. In [19] Rammaha extended this result to the 2D case. For the Euler-Poisson system, Guo and Tahvildar-Zadeh [10] established a "Siderian" blowup result for spherically symmetric initial data. Recently Chae and Tadmor [2] proved finite-time blow-up for $C^1$ solutions of a class of pressureless attractive Euler-Poisson equations in $\mathbb{R}^n$, $n \geq 1$. These negative results showed the abundance of shock waves for large solutions.

The "Poisson"-part of the Euler-Poisson system has a stabilizing effect which makes the whole analysis of (1.1) quite different from the pure compressible Euler equations. This is best understood in analyzing small irrotational perturbations of the equilibrium state $n \equiv n_0$, $\mathbf{u} \equiv 0$. For the 3D compressible Euler equation with irrotational initial data $(n_\epsilon(0), \mathbf{u}_\epsilon(0)) = (\epsilon\rho_0 + n_0, \epsilon\mathbf{v}_0)$, where $\rho_0 \in \mathcal{S}(\mathbb{R}^3)$, $\mathbf{v}_0 \in \mathcal{S}(\mathbb{R}^3)^3$ are fixed functions ($\epsilon$ sufficiently small), Sideris [22] proved that the lifespan of the classical solution $T_\epsilon > \exp(C/\epsilon)$. For the upper bound it follows from his previous paper [21] that $T_\epsilon < \exp(C/\epsilon^2)$. Sharper results are obtained by Godin [7] in which he show for radial initial data as a smooth compact $\epsilon$-perturbation of the constant state, the precise asymptotic of the lifespan $T_\epsilon$ is exponential in the sense

$$\lim_{\epsilon\to 0+} \epsilon \log T_\epsilon = T^*,$$

where $T^*$ is a constant. All these results rely crucially on the observation that after some simple reductions, the compressible Euler equation in rescaled variables is given by a vectorial nonlinear wave equation with pure quadratic nonlinearities. The linear part of the wave equation decays at most at the speed $t^{-(d-1)/2}$ which in 3D is not integrable. Unless the nonlinearity has some additional nice structure such as the null condition [1, 15], one cannot in general expect global existence of small solutions. On the other hand, the situation for the Euler-Poisson system (1.1) is quite different due to the additional Poisson coupling term. As was already explained before, the Euler-Poisson system (1.1) expressed in rescaled variables is given by the quasi-linear Klein-Gordon system (1.6) for which the linear solutions have an enhanced decay of $(1 + t)^{-d/2}$. This is in sharp contrast with the pure

Euler case for which the decay is only $t^{-(d-1)/2}$. Note that in $d=3$, $(1+t)^{-d/2}=(1+t)^{-3/2}$ which is integrable in $t$. In a seminal paper [8], by exploiting the crucial decay property of the Klein-Gordon flow in 3D, Guo [8] modified Shatah's normal form method [20] and constructed a smooth irrotational global solution to (1.1) around the equilibrium state $(n_0,0)$ for which the perturbations decay at a rate $C_p\cdot(1+t)^{-p}$ for any $1<p<3/2$ (here $C_p$ denotes a constant depending on the parameter $p$). Note in particular that the sharp decay $t^{-3/2}$ is marginally missed here due to a technical complication caused by the nonlocal Riesz operator in the nonlinearity.

Construction of smooth global solutions to (1.1) in the two-dimensional case was open since Guo's work. The first obstacle comes from slow dispersion since the linear solution to the Klein-Gordon system in $d=2$ decays only at $(1+t)^{-1}$ which is not integrable, in particular making the strategy of [8] difficult to apply. The other main technical difficulty comes from the nonlocal nonlinearity in (1.6) which involves a Riesz-type singular operator. For general scalar quasi-linear Klein-Gordon equations in 3D with quadratic type nonlinearities, global small smooth solutions were first constructed independently by Klainerman [14] using the invariant vector field method and Shatah [20] using a normal form method. Even in 3D there are essential technical difficulties in employing Klainerman's invariant vector field method due to the Riesz type nonlocal term in (1.6). The Klainerman invariant vector fields consist of infinitesimal generators which commute well with the linear operator $\partial_{tt}-\Delta+1$. The most problematic part comes from the Lorentz boost $\Omega_{0j}=t\partial_{x_j}+x_j\partial_t$. While the first part $t\partial_{x_j}$ commutes naturally with the Riesz operator $R_{ij}=(-\Delta)^{-1}\partial_{x_i}\partial_{x_j}$, the second part $x_j\partial_t$ interacts rather badly with $R_{ij}$, producing a commutator which scales as

$$[x_j\partial_t, R_{ij}]\sim\partial_t|\nabla|^{-1}.$$

After repeated commutation of these operators one obtain in general terms of the form $|\nabla|^{-N}$ which makes the low frequency part of the solution out of control. It is for this reason that in 3D case Guo [8] adopted Shatah's method of normal form in $L^p$ $(p>1)$ setting for which the Riesz term $R_{ij}$ causes no trouble. We turn now to the 2D Klein-Gordon equations with pure quadratic nonlinearities. In this case, direct applications of either Klainerman's invariant vector field method or Shatah's normal form method are not possible since the linear solutions only decay at a speed of $(1+t)^{-1}$ which is not integrable and makes the quadratic nonlinearity quite resonant. In [23], Simon and Taflin constructed wave operators for the 2D semilinear Klein-Gordon system with quadratic nonlinearities. In [18], Ozawa, Tsutaya and Tsutsumi considered the Cauchy problem and constructed smooth global solutions by first transforming the quadratic nonlinearity into a cubic one using Shatah's normal form method and then applying Klainerman's invariant vector field method to obtain decay of intermediate norms. Due to the nonlocal complication with the Lorentz boost which we explained earlier, this approach seems difficult to apply in the 2D Euler-Poisson system.

As was already mentioned, the purpose of this work is to settle the Cauchy problem for (1.1) in the two-dimensional case. Before we state our main results, we need to make some further simplifications. Since $\mathbf{v}$ is irrotational, we can write

$\mathbf{v} = \nabla\phi_1$ and obtain from (1.5):

$$\begin{cases} \partial_t u + \Delta\phi_1 + \nabla\cdot(u\nabla\phi_1) = 0, \\ \partial_t\phi_1 + |\nabla|^{-2}\langle\nabla\rangle^2 u + \frac{1}{2}(u^2 + |\nabla\phi_1|^2) = 0. \end{cases} \tag{1.7}$$

We can diagonalize the system (1.7) by introducing the complex scalar function

$$\begin{aligned} h(t) &= \frac{\langle\nabla\rangle}{|\nabla|}u - i|\nabla|\phi_1 \\ &= \frac{\langle\nabla\rangle}{|\nabla|}u + i\frac{\nabla}{|\nabla|}\cdot\mathbf{v}. \end{aligned} \tag{1.8}$$

Note that since $\mathbf{v}$ is irrotational, we have

$$\mathbf{v} = -\frac{\nabla}{|\nabla|}\mathrm{Im}(h). \tag{1.9}$$

By (1.5), we have

$$\begin{aligned} h(t) = e^{it\langle\nabla\rangle}h_0 + \int_0^t e^{i(t-s)\langle\nabla\rangle}\Big(&-\frac{\langle\nabla\rangle\nabla}{|\nabla|}\cdot(u\mathbf{v}) \\ &+ \frac{i}{2}|\nabla|(u^2 + |\mathbf{v}|^2)\Big)ds, \end{aligned} \tag{1.10}$$

where $h_0$ is the initial data given by

$$h_0 = \frac{\langle\nabla\rangle}{|\nabla|}u_0 + i\frac{\nabla}{|\nabla|}\cdot\mathbf{v}_0.$$

Here $u_0$ is the initial density (perturbation) and $\mathbf{v}_0$ is the initial velocity.

We need the following

**Choice of constants**: we shall choose positive parameters $1 \ll N_1 \ll N' \ll N$, $\epsilon_1 \ll \frac{\delta_2}{N_1} \ll \delta_2 \ll 1$, $\delta_1 \ll \frac{\delta_2}{N_1} \ll \delta_2 \ll 1$, and $N \gg \frac{1}{\delta_1}$. Let $q = N_1/\epsilon_1$.

*Remark* 1.1. Here the parameter $N_1$ serves the role of a large but universal constant. The small constant $\delta_2$ is only needed in intermediate estimates but not in the final statement of our theorem.

Now introduce the norms

$$\begin{aligned} \|h\|_X := &\|\langle t\rangle|\nabla|^{\frac{1}{2}}\langle\nabla\rangle h(t)\|_{L_t^\infty L_x^\infty} + \|\langle t\rangle^{-\delta_1}h(t)\|_{L_t^\infty H_x^N} \\ &+ \|\langle t\rangle^{1-\frac{2}{q}}h(t)\|_{L_t^\infty L_x^q} + \|h(t)\|_{L_t^\infty H_x^{N'}} + \|\langle x\rangle e^{-it\langle\nabla\rangle}h(t)\|_{L_t^\infty L_x^{2+\epsilon_1}}; \end{aligned} \tag{1.11}$$

and

$$\begin{aligned} \|h\|_{X_1} := &\|\langle t\rangle|\nabla|^{\frac{1}{2}}\langle\nabla\rangle h(t)\|_{L_t^\infty L_x^\infty} + \|\langle t\rangle^{1-\frac{2}{q}}h(t)\|_{L_t^\infty L_x^q} + \|h(t)\|_{L_t^\infty H_x^{N'}} \\ &+ \|\langle x\rangle e^{-it\langle\nabla\rangle}h(t)\|_{L_t^\infty L_x^{2+\epsilon_1}}. \end{aligned} \tag{1.12}$$

Note that $\|h\|_{X_1}$ collects all intermediate norms of $h$ except the highest energy norm $\|h\|_{H^N}$. This is because for the intermediate norms we have to use the normal formal technique and perform a fine decomposition of the solution which in turn can be estimated separately.

Here we choose the norm $\||\nabla|^{\frac{1}{2}}\langle\nabla\rangle h(t)\|_{L_x^\infty}$ for simplicity. Actually $\||\nabla|^s\langle\nabla\rangle h(t)\|_{L_x^\infty}$ for some $0 < s < 1$ would also suffice for our estimates. However we cannot take $s = 0$ due to the non-locality caused by the Riesz transform.

Our result is expressed in the following

**Theorem 1.2** (Smooth global solutions for the Cauchy problem)**.** *There exists $\epsilon > 0$ sufficiently small such that if initial data $h_0$ is regular enough (for example $\|e^{it\langle\nabla\rangle}h_0\|_X \leq \epsilon$ ), then there exists a unique smooth global solution to the 2D Euler-Poisson system* (1.8)–(1.10)*. Moreover the solution scatters in the energy space $H^{N'}$.*

*Remark* 1.3. We did not make any effort to optimize the assumptions on the initial data or the function space that is used in the analysis. The whole point is to construct *smooth* global in time classical solutions.

The rest of this paper is organized as follows. In Section 2 we gather some preliminary linear estimates. In Section 3 we perform some preliminary transformations and decompose the solution into three parts: the initial data, the boundary term $g$ and the cubic interaction term $f_{\text{cubic}}$. Section 4 is devoted to the estimate of the boundary term $g$. In Section 5 we control the high frequency part of cubic interactions. In Section 6 we control the low frequency part of cubic interactions which is the most delicate part of our analysis. In Section 7 we complete the proof of our main theorem.

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## 2. Preliminaries

2.1. **Some notations.** We write $X \lesssim Y$ or $Y \gtrsim X$ to indicate $X \leq CY$ for some constant $C > 0$. We use $O(Y)$ to denote any quantity $X$ such that $|X| \lesssim Y$. We use the notation $X \sim Y$ whenever $X \lesssim Y \lesssim X$. The fact that these constants depend upon the dimension $d$ will be suppressed. If $C$ depends upon some additional parameters, we will indicate this with subscripts; for example, $X \lesssim_u Y$ denotes the assertion that $X \leq C_u Y$ for some $C_u$ depending on $u$. Sometimes when the context is clear, we will suppress the dependence on $u$ and write $X \lesssim_u Y$ as $X \lesssim Y$. We will write $C = C(Y_1, \cdots, Y_n)$ to stress that the constant $C$ depends on quantities $Y_1, \cdots, Y_n$. We denote by $X\pm$ any quantity of the form $X \pm \epsilon$ for any $\epsilon > 0$.

We use the 'Japanese bracket' convention $\langle x\rangle := (1 + |x|^2)^{1/2}$.

We write $L_t^q L_x^r$ to denote the Banach space with norm

$$\|u\|_{L_t^q L_x^r(\mathbb{R}\times\mathbb{R}^d)} := \Big(\int_{\mathbb{R}} \Big(\int_{\mathbb{R}^d} |u(t,x)|^r \, dx\Big)^{q/r} \, dt\Big)^{1/q},$$

with the usual modifications when $q$ or $r$ are equal to infinity, or when the domain $\mathbb{R}\times\mathbb{R}^d$ is replaced by a smaller region of spacetime such as $I \times \mathbb{R}^d$. When $q = r$ we abbreviate $L_t^q L_x^q$ as $L_{t,x}^q$.

We will use $\phi \in C^\infty(\mathbb{R}^d)$ be a radial bump function supported in the ball $\{x \in \mathbb{R}^d : |x| \leq \frac{25}{24}\}$ and equal to one on the ball $\{x \in \mathbb{R}^d : |x| \leq 1\}$. For any constant $C > 0$, we denote $\phi_{\leq C}(x) := \phi\big(\frac{x}{C}\big)$ and $\phi_{>C} := 1 - \phi_{\leq C}$. We also denote $\chi_{>C} = \phi_{>C}$ sometimes.

We will often need the Fourier multiplier operators defined by the following:

$$\mathcal{F}\Big(T_{m(\xi,\eta)}(f,g)\Big)(\xi) = \int m(\xi,\eta)\hat{f}(\xi-\eta)\hat{g}(\eta)d\eta,$$

$$\mathcal{F}\Big(T_{m(\xi,\eta,\sigma)}(f,g,h)\Big)(\xi) = \int m(\xi,\eta,\sigma)\hat{f}(\xi-\eta)\hat{g}(\eta-\sigma)\hat{h}(\sigma)d\eta d\sigma.$$

Similarly one can define $T_m(f_1,\cdots,f_n)$ for functions $f_1,\cdots,f_n$ and a general symbol $m = m(\xi,\eta_1,\cdots,\eta_{n-1})$.

2.2. **Basic harmonic analysis.** For each number $N > 0$, we define the Fourier multipliers

$$\widehat{P_{\leq N} f}(\xi) := \phi_{\leq N}(\xi)\hat{f}(\xi)$$
$$\widehat{P_{> N} f}(\xi) := \phi_{> N}(\xi)\hat{f}(\xi)$$
$$\widehat{P_N f}(\xi) := (\phi_{\leq N} - \phi_{\leq N/2})(\xi)\hat{f}(\xi)$$

and similarly $P_{<N}$ and $P_{\geq N}$. We also define

$$P_{M<\cdot\leq N} := P_{\leq N} - P_{\leq M} = \sum_{M<N'\leq N} P_{N'}$$

whenever $M < N$. We will usually use these multipliers when $M$ and $N$ are *dyadic numbers* (that is, of the form $2^n$ for some integer $n$); in particular, all summations over $N$ or $M$ are understood to be over dyadic numbers. Nevertheless, it will occasionally be convenient to allow $M$ and $N$ to not be a power of 2. As $P_N$ is not truly a projection, $P_N^2 \neq P_N$, we will occasionally need to use fattened Littlewood-Paley operators:

$$\tilde{P}_N := P_{N/2} + P_N + P_{2N}. \tag{2.1}$$

These obey $P_N\tilde{P}_N = \tilde{P}_N P_N = P_N$.

Like all Fourier multipliers, the Littlewood-Paley operators commute with the propagator $e^{it\Delta}$, as well as with differential operators such as $i\partial_t + \Delta$. We will use basic properties of these operators many times, including

**Lemma 2.1** (Bernstein estimates)**.** *For* $1 \leq p \leq q \leq \infty$,

$$\big\||\nabla|^{\pm s} P_M f\big\|_{L_x^p(\mathbb{R}^d)} \sim M^{\pm s}\|P_M f\|_{L_x^p(\mathbb{R}^d)},$$
$$\|P_{\leq M} f\|_{L_x^q(\mathbb{R}^d)} \lesssim M^{\frac{d}{p}-\frac{d}{q}}\|P_{\leq M} f\|_{L_x^p(\mathbb{R}^d)},$$
$$\|P_M f\|_{L_x^q(\mathbb{R}^d)} \lesssim M^{\frac{d}{p}-\frac{d}{q}}\|P_M f\|_{L_x^p(\mathbb{R}^d)}.$$

We shall use the following lemma several times which allows us to commute the $L^p$ estimates with the linear flow $e^{it\langle\nabla\rangle}$. Roughly speaking it says that for $t \gtrsim 1$,

$$\|P_{<t^C} e^{it\langle\nabla\rangle} f\|_p \lesssim t^{0+}\|f\|_p, \quad p = 2+ \text{ or } p = 2- .$$

**Lemma 2.2.** *For any* $1 \leq p \leq \infty$*,* $t \geq 0$ *and dyadic* $M > 0$*, we have*

$$\|e^{it\langle\nabla\rangle} P_{<M} g\|_p \lesssim \langle Mt\rangle^{|1-\frac{2}{p}|}\|g\|_p. \tag{2.2}$$

*Also for any* $1 \leq p \leq \infty$*,* $t \geq 0$*, we have*

$$\|e^{it\langle\nabla\rangle} g\|_p \lesssim \langle t\rangle^{|1-\frac{2}{p}|}\|\langle\nabla\rangle^4 g\|_p. \tag{2.3}$$

*Proof.* We shall only prove (2.2). The proof of (2.3) is similar and therefore omitted. The idea is to use interpolation between $p = 1$, $p = 2$ and $p = \infty$. We consider only the case $p = \infty$. The other case $p = 1$ is similar. To establish the inequality it suffices to bound the $L^1_x$ norm of the kernel $e^{it\langle\nabla\rangle}P_{<M}$.

Note that $e^{it\langle\nabla\rangle}P_{<M}f = K * f$, where

$$\hat{K}(\xi) = e^{it\langle\xi\rangle}\phi(\frac{\xi}{M}).$$

Observe $\|K\|_{L^2_x} \lesssim M$ and for $t > 0$,

$$\||x|^2K(x)\|_{L^2_x} = \|\partial^2_\xi(\hat{K}(\xi))\|_{L^2_\xi} \lesssim t^2M + t + \frac{1}{M}.$$

Then

$$\|K\|_{L^1_x} \lesssim \|K\|^{\frac{1}{2}}_{L^2_x}\||x|^2K\|^{\frac{1}{2}}_{L^2_x} \lesssim \langle Mt\rangle.$$

The desired inequality then follows from Young's inequality. □

Finally we need some simple lemma from vector algebra.

**Lemma 2.3.** *For any $x \in \mathbb{R}^2$, $y \in \mathbb{R}^2$, we have*

$$\left|\frac{x}{\langle x\rangle} - \frac{y}{\langle y\rangle}\right| \gtrsim \frac{|x-y|}{\langle |x|+|y|\rangle^3}. \tag{2.4}$$

*More over there exists a matrix $Q = Q(x,y)$ such that*

$$\frac{x}{\langle x\rangle} - \frac{y}{\langle y\rangle} = Q(x,y)(x-y) \tag{2.5}$$

*Also*

$$\frac{1}{\langle |x|+|y|\rangle^3} \lesssim \|Q(x,y)\| \lesssim 1,$$

*Proof.* The first inequality is the essentially Lemma (4.3) in [12]. Or one can prove it directly. The last two properties are simple consequences of matrix norms. □

We shall need to exploit some subtle cancelations of the phases. The following lemma will be useful in our nonlinear estimates.

**Lemma 2.4.** *Consider the following phases:*

$$\phi_1(\xi,\eta,\sigma) = \langle\xi\rangle + \langle\xi-\eta\rangle - \langle\eta-\sigma\rangle - \langle\sigma\rangle,$$
$$\phi_2(\xi,\eta,\sigma) = \langle\xi\rangle - \langle\xi-\eta\rangle + \langle\eta-\sigma\rangle - \langle\sigma\rangle,$$
$$\phi_3(\xi,\eta,\sigma) = \langle\xi\rangle - \langle\xi-\eta\rangle - \langle\eta-\sigma\rangle + \langle\sigma\rangle.$$

*There exist smooth matrix functions $Q_{11} = Q_{11}(\xi,\eta,\sigma)$, $Q_{12} = Q_{12}(\xi,\eta,\sigma)$, $Q_{21} = Q_{21}(\xi,\eta)$, $Q_{22} = Q_{22}(\eta,\sigma)$, $Q_{31} = Q_{31}(\xi,\eta)$, $Q_{32} = Q_{32}(\eta,\sigma)$ such that*

$$\partial_\xi\phi_1 = Q_{11}(\xi,\eta,\sigma)\partial_\eta\phi_1 + Q_{12}(\xi,\eta,\sigma)\partial_\sigma\phi_1,$$
$$\partial_\xi\phi_2 = Q_{21}(\xi,\eta)Q_{22}(\eta,\sigma)\partial_\sigma\phi_2,$$
$$\partial_\xi\phi_3 = Q_{31}(\xi,\eta)Q_{32}(\eta,\sigma)\partial_\sigma\phi_3.$$

*Moreover we have the point-wise (polynomial) bounds*

$$
\begin{aligned}
|\partial_\xi^\alpha \partial_\eta^\beta \partial_\sigma^\gamma Q_{11}(\xi,\eta,\sigma)| + |\partial_\xi^\alpha \partial_\eta^\beta \partial_\sigma^\gamma Q_{12}(\xi,\eta,\sigma)| &\lesssim \langle |\xi|+|\eta|+|\sigma|\rangle^{C(|\alpha|+|\beta|+|\gamma|+1)},\\
|\partial_\xi^\alpha \partial_\eta^\beta Q_{21}(\xi,\eta)| + |\partial_\xi^\alpha \partial_\eta^\beta Q_{31}(\xi,\eta)| &\lesssim \langle |\xi|+|\eta|\rangle^{C(|\alpha|+|\beta|+1)},\\
|\partial_\eta^\alpha \partial_\sigma^\beta Q_{22}(\eta,\sigma)| + |\partial_\eta^\alpha \partial_\sigma^\beta Q_{32}(\eta,\sigma)| &\lesssim \langle |\eta|+|\sigma|\rangle^{C(|\alpha|+|\beta|+1)}.
\end{aligned}
\tag{2.6}
$$

*Proof.* We prove it for $\phi_1$. The other two cases are simpler. By Lemma 2.3, we write

$$
\begin{aligned}
\partial_\xi \phi_1 &= \frac{\xi}{\langle \xi\rangle} + \frac{\xi-\eta}{\langle \xi-\eta\rangle} = \tilde{Q}_1\cdot(2\xi-\eta),\\
\partial_\eta \phi_1 &= \frac{\eta-\xi}{\langle \eta-\xi\rangle} - \frac{\eta-\sigma}{\langle \eta-\sigma\rangle} = \tilde{Q}_2\cdot(\xi-\sigma),\\
\partial_\sigma \phi_1 &= \frac{\eta-\sigma}{\langle \eta-\sigma\rangle} - \frac{\sigma}{\langle \sigma\rangle} = \tilde{Q}_3\cdot(\eta-2\sigma).
\end{aligned}
$$

Hence

$$
\begin{aligned}
\partial_\xi \phi_1 &= \tilde{Q}_1\left(2\tilde{Q}_2^{-1}\partial_\eta \phi_1 - \tilde{Q}_3^{-1}\partial_\sigma\phi_1\right)\\
&=: Q_{11}\partial_\eta\phi_1 + Q_{12}\partial_\sigma\phi_1.
\end{aligned}
$$

The bound (2.6) is obvious. □

## 3. Preliminary transformations

Since the function $h = h(t,x)$ is complex-valued, we write it as

$$h(t,x) = h_1(t,x) + ih_2(t,x).$$

By (1.8) and (1.9), we have

$$
\begin{aligned}
u &= \frac{|\nabla|}{\langle\nabla\rangle}h_1,\\
\mathbf{v} &= -\frac{\nabla}{|\nabla|}h_2.
\end{aligned}
$$

In Fourier space, (1.10) then takes the form

$$
\begin{aligned}
&\hat{h}(t,\xi)\\
=&e^{it\langle\xi\rangle}\widehat{h_0}(\xi) - \int_0^t\int e^{i(t-s)\langle\xi\rangle}\langle\xi\rangle\langle\eta\rangle^{-1}|\eta|\frac{\xi\cdot(\xi-\eta)}{|\xi||\xi-\eta|}\widehat{h_1}(s,\eta)\widehat{h_2}(s,\xi-\eta)d\eta ds\\
&+\frac{i}{2}\int_0^t\int e^{i(t-s)\langle\xi\rangle}|\xi|\frac{|\eta||\xi-\eta|}{\langle\eta\rangle\langle\xi-\eta\rangle}\widehat{h_1}(s,\eta)\widehat{h_1}(s,\xi-\eta)d\eta ds\\
&-\frac{i}{2}\int_0^t\int e^{i(t-s)\langle\xi\rangle}|\xi|\frac{\eta\cdot(\xi-\eta)}{|\eta||\xi-\eta|}\widehat{h_2}(s,\eta)\widehat{h_2}(s,\xi-\eta)d\eta ds.
\end{aligned}
$$

Denote

$$f(t) = e^{-it\langle\nabla\rangle}h(t).$$

Then after a tedious calculation,

$$\begin{aligned}
\hat{f}(t,\xi) = \hat{h}_0(\xi) &+ \int_0^t \int e^{-is(\langle\xi\rangle-\langle\eta\rangle-\langle\xi-\eta\rangle)}\Big(\frac{i}{4}\langle\xi\rangle\langle\eta\rangle^{-1}|\eta|\frac{\xi\cdot(\xi-\eta)}{|\xi||\xi-\eta|} \\
&\qquad + \frac{i}{8}|\xi|\frac{|\eta||\xi-\eta|}{\langle\eta\rangle\langle\xi-\eta\rangle} + \frac{i}{8}|\xi|\frac{\eta\cdot(\xi-\eta)}{|\eta||\xi-\eta|}\Big)\hat{f}(s,\eta)\hat{f}(s,\xi-\eta)d\eta ds \\
&+ \int_0^t \int e^{-is(\langle\xi\rangle-\langle\eta\rangle+\langle\xi-\eta\rangle)}\Big(-\frac{i}{4}\langle\xi\rangle\langle\eta\rangle^{-1}|\eta|\frac{\xi\cdot(\xi-\eta)}{|\xi||\xi-\eta|} \\
&\qquad + \frac{i}{8}|\xi|\frac{|\eta||\xi-\eta|}{\langle\eta\rangle\langle\xi-\eta\rangle} - \frac{i}{8}|\xi|\frac{\eta\cdot(\xi-\eta)}{|\eta||\xi-\eta|}\Big)\hat{f}(s,\eta)\overline{\hat{f}(s,\eta-\xi)}d\eta ds \\
&+ \int_0^t \int e^{-is(\langle\xi\rangle+\langle\eta\rangle-\langle\xi-\eta\rangle)}\Big(\frac{i}{4}\langle\xi\rangle\langle\eta\rangle^{-1}|\eta|\frac{\xi\cdot(\xi-\eta)}{|\xi||\xi-\eta|} \\
&\qquad + \frac{i}{8}|\xi|\frac{|\eta||\xi-\eta|}{\langle\eta\rangle\langle\xi-\eta\rangle} - \frac{i}{8}|\xi|\frac{\eta\cdot(\xi-\eta)}{|\eta||\xi-\eta|}\Big)\overline{\hat{f}(s,-\eta)}\hat{f}(s,\xi-\eta)d\eta ds \\
&+ \int_0^t \int e^{-is(\langle\xi\rangle+\langle\eta\rangle+\langle\xi-\eta\rangle)}\Big(-\frac{i}{4}\langle\xi\rangle\langle\eta\rangle^{-1}|\eta|\frac{\xi\cdot(\xi-\eta)}{|\xi||\xi-\eta|} \\
&\qquad + \frac{i}{8}|\xi|\frac{|\eta||\xi-\eta|}{\langle\eta\rangle\langle\xi-\eta\rangle} + \frac{i}{8}|\xi|\frac{\eta\cdot(\xi-\eta)}{|\eta||\xi-\eta|}\Big)\overline{\hat{f}(s,-\eta)}\ \overline{\hat{f}(s,\eta-\xi)}d\eta ds \qquad (3.1)
\end{aligned}$$

Here $\overline{\hat{f}}$ denote the complex conjugate of $\hat{f}$. Note that

$$\begin{aligned}
\overline{\hat{f}(t,-\xi)} &= e^{it\langle\xi\rangle}\hat{\bar{h}}(t,\xi), \\
\hat{f}(t,\xi) &= e^{-it\langle\xi\rangle}\hat{h}(t,\xi).
\end{aligned}$$

To simplify matters, we shall write (3.1) as

$$\hat{f}(t,\xi) = \hat{h}_0(\xi) + \int_0^t \int e^{-is\phi(\xi,\eta)}m_0(\xi,\eta)\hat{f}(s,\xi-\eta)\hat{f}(s,\eta)d\eta ds, \qquad (3.2)$$

where

$$\phi(\xi,\eta) = \langle\xi\rangle \pm \langle\xi-\eta\rangle \pm \langle\eta\rangle, \qquad (3.3)$$

and $m_0(\xi,\eta)$ is given by (after some symmetrization between $\eta$ and $\xi-\eta$)

$$\begin{aligned}
m_0(\xi,\eta) &= const\cdot\langle\xi\rangle\frac{\xi\cdot\eta}{|\xi||\eta|}\frac{|\xi-\eta|}{\langle\xi-\eta\rangle} + const\cdot\langle\xi\rangle\frac{\xi\cdot(\xi-\eta)}{|\xi||\xi-\eta|}\frac{|\eta|}{\langle\eta\rangle} \\
&\qquad + const\cdot|\xi|\cdot\frac{|\eta|}{\langle\eta\rangle}\cdot\frac{|\xi-\eta|}{\langle\xi-\eta\rangle} + const\cdot|\xi|\frac{(\xi-\eta)\cdot\eta}{|\xi-\eta||\eta|} \\
&:= \sum_{i=1}^4 m_i(\xi,\eta).
\end{aligned}$$

Here and in the rest of this paper we shall abuse slightly the notation and denote $\hat{f}(t,\xi)$ to be either itself or its complex conjugate (i.e. $\overline{\hat{f}(t,-\xi)}$, see (3.1)). Note that in the expression of $m_0(\xi,\eta)$ there are four types of symbols. We write them collectively as

$$m_0(\xi,\eta) = \sum_{1\le j,k,l\le 2} a_{jkl}(\xi,\eta)\frac{\xi_j}{|\xi|}\frac{\eta_k}{|\eta|}\frac{\xi_l-\eta_l}{|\xi-\eta|}, \qquad (3.4)$$

where $a_{jkl}(\xi,\eta) \in C^\infty$ and has the bound

$$|\partial_\eta^\alpha \partial_\xi^\beta a_{jkl}(\xi,\eta)| \lesssim_{\alpha,\beta} \langle \xi \rangle, \qquad \forall\, \xi, \eta.$$

For example we can write $m_2(\xi,\eta)$ as

$$m_2(\xi,\eta) = \langle \xi \rangle \langle \eta \rangle^{-1} \eta \cdot \frac{\eta}{|\eta|} \frac{\xi \cdot (\xi-\eta)}{|\xi||\xi-\eta|}.$$

Although the frequency variables $(\xi,\eta)$ are vectors, this fact will play no role in our analysis. The explicit form of $a_{jkl}$ in (3.4) will also not be important. Therefore we shall suppress the subscript notations and summation in (3.4), pretend everything is scalar valued and write $m_0(\xi,\eta)$ as a general symbol

$$m_0(\xi,\eta) = a(\xi,\eta) \frac{\xi}{|\xi|} \frac{\xi-\eta}{|\xi-\eta|} \frac{\eta}{|\eta|}, \tag{3.5}$$

where $a \in C^\infty$ and has polynomially bounded derivatives in the sense that:

$$|\partial_\xi^\alpha \partial_\eta^\beta a(\xi,\eta)| \lesssim_{\alpha,\beta} \langle \xi \rangle. \tag{3.6}$$

One should think of $m_0(\xi,\eta)$ as any one of the summand in (3.4). Observe $m_0(\xi,\eta)$ is symmetric in the sense that

$$m_0(\xi,\eta) = m_0(\xi,\xi-\eta). \tag{3.7}$$

The nice feature of Klein-Gordon is

$$|\phi(\xi,\eta)| \gtrsim 1/\langle |\xi| + |\eta| \rangle, \qquad \text{for any } (\xi,\eta).$$

We can then integrate by parts in the time variable $s$ in (3.2). By (3.7),

$$\begin{aligned} &\int_0^t \int e^{-is\phi(\xi,\eta)} \frac{m_0(\xi,\eta)}{\phi(\xi,\eta)} \partial_s \hat{f}(s,\xi-\eta) \hat{f}(s,\eta) d\eta \\ =&\int_0^t \int e^{-is\phi(\xi,\eta)} \frac{m_0(\xi,\eta)}{\phi(\xi,\eta)} \partial_s \hat{f}(s,\eta) \hat{f}(s,\xi-\eta) d\eta \end{aligned} \tag{3.8}$$

using the change of variable $\eta \to \xi - \eta$. In the above equality we have abused again the notation and denote $\phi(\xi,\eta) = \phi(\xi,\xi-\eta)$ since it will remain of the same form as (3.3).

Integrating by parts in the time variable $s$ in (3.2) and using (3.8), we obtain

$$\begin{aligned} \hat{f}(t,\xi) &= \widehat{\tilde{h}_0}(\xi) + \hat{g}(t,\xi) \\ &\quad + \int_0^t \int e^{-is\phi(\xi,\eta,\sigma)} m_1(\xi,\eta,\sigma) \hat{f}(s,\xi-\eta) \hat{f}(s,\eta-\sigma) \hat{f}(s,\sigma) d\sigma d\eta ds, \\ &=: \widehat{\tilde{h}_0}(\xi) + \hat{g}(t,\xi) + \hat{f}_{\mathrm{cubic}}(t,\xi), \end{aligned} \tag{3.9}$$

where $\tilde{h}_0$ collects the contribution from the boundary term $s=0$ and data $h_0$:

$$\widehat{\tilde{h}_0}(\xi) = \hat{h}_0(\xi) + \int \frac{m_0(\xi,\eta)}{i\phi(\xi,\eta)} \hat{h}_0(\xi-\eta) \hat{h}_0(\eta) d\eta; \tag{3.10}$$

the term $g$ denotes the boundary term arising from $s=t$:

$$\hat{g}(t,\xi) = \int e^{-it\phi(\xi,\eta)} \cdot \frac{m_0(\xi,\eta)}{-i\phi(\xi,\eta)} \hat{f}(t,\xi-\eta) \hat{f}(t,\eta) d\eta; \tag{3.11}$$

$m_1(\xi,\eta,\sigma)$ is given by (here we use (3.7))

$$m_1(\xi,\eta,\sigma)=\frac{m_0(\xi,\eta)m_0(\eta,\sigma)}{\phi(\xi,\eta)};$$

and also

$$\phi(\xi,\eta,\sigma)=\langle\xi\rangle\pm\langle\xi-\eta\rangle\pm\langle\eta-\sigma\rangle\pm\langle\sigma\rangle.$$

## 4. Estimates of the boundary term $g$

In this section we control the boundary term $g$ coming from integration by parts in the time variable $s$ (see (3.11)).

We have the following

**Proposition 4.1.**

$$\|g\|_{L_t^\infty H_x^{N'}}+\|\langle t\rangle g\|_{L_t^\infty H_x^{\frac{N'}{2}}}+\|\langle x\rangle g\|_{L_t^\infty L_x^{2+\epsilon_1}}\lesssim\|h\|_X^2.$$

By Proposition 4.1 and Sobolev embedding, it is easy to prove the following

**Corollary 4.2.**

$$\|e^{it\langle\nabla\rangle}g\|_{X_1}\lesssim\|h\|_X^2.$$

*Remark* 4.3. Note that in Proposition 4.1 the $H^{N'/2}$ norm of $g$ decays at a rate $1/\langle t\rangle$. This is not surprising since $g$ essentially is the quadratic interaction of linear waves.

The proof of Proposition 4.1 will occupy the rest of this section. We begin with the $H^{N'}$ estimate.

By (3.5), we can write $g$ as

$$\hat g(t,\xi)=\frac{\xi}{|\xi|}\left(\int e^{-it\phi(\xi,\eta)}\tilde a(\xi,\eta)\widehat{\mathcal{R}f}(t,\xi-\eta)\widehat{\mathcal{R}f}(t,\eta)d\eta\right),$$

where $\tilde a(\xi,\eta)\in C^\infty$ and has the bound

$$|\partial_\xi^\alpha\partial_\eta^\beta\tilde a(\xi,\eta)|\lesssim_{\alpha,\beta}(\langle|\xi|+|\eta|\rangle)^{1+|\alpha|+|\beta|}.$$

Recall the parameters $1\ll N_1\ll N'\ll N$. Clearly

$$\begin{aligned}\|g(t)\|_{H^{N'}}&=\|T_{\tilde a(\xi,\eta)\langle\xi-\eta\rangle^{-N_1}\langle\eta\rangle^{-N_1}}(\langle\nabla\rangle^{N_1}\mathcal{R}h,\langle\nabla\rangle^{N_1}\mathcal{R}h)\|_{H^{N'}}\\&\lesssim\|\langle\nabla\rangle^{N'+100N_1}h\|_4^2\\&\lesssim\|h(t)\|_{H^N}^{\frac23}\|h(t)\|_8^{\frac43}\\&\lesssim\langle t\rangle^{\frac23\delta_1-1}\|h\|_X^2\\&\lesssim\|h\|_X^2\end{aligned}$$

which is clearly good for us.

By a similar calculation, we also obtain

$$\begin{aligned}&\|g(t)\|_{H^{N'/2}}\\\lesssim&\|h(t)\|_{H^{N'}}^{\frac23}\|h(t)\|_8^{\frac43}\\\lesssim&\langle t\rangle^{-1}\|h\|_X^2\end{aligned}$$

which is also good.

It remains for us to show the estimate

$$\|\langle x\rangle g(t)\|_{2+\epsilon_1} \lesssim \|h\|_X^2.$$

For this we will have to introduce some frequency cut-offs. The main idea is that in the high frequency regime we use smoothing (from $H^N$-energy) and in the low frequency regime we use directly dispersive bounds. To simplify the presentation, we divide the proof into two lemmas. Consider first the low frequency piece.

Let

$$\hat{g}_0(t,\xi) = \frac{\xi}{|\xi|}\int e^{-it\phi(\xi,\eta)} a_0(\xi,\eta)\widehat{\mathcal{R}f}(t,\xi-\eta)\widehat{\mathcal{R}f}(t,\eta)d\eta,$$

where

$$a_0(\xi,\eta) = \tilde{a}(\xi,\eta)\chi_{|\xi-\eta|\lesssim\langle t\rangle^{\delta_1}}\chi_{|\eta|\lesssim\langle t\rangle^{\delta_1}}.$$

Note that the symbol $a_0(\xi,\eta)$ is restricted to the non-high frequency regime $\lesssim \langle t\rangle^{0+}$.

**Lemma 4.4.** *We have the estimate*

$$\|\langle x\rangle g_0\|_{2+\epsilon_1} \lesssim \|h\|_X^2.$$

*Proof.* Write $g_0 = \mathcal{R}g_1$, where $R$ is the Riesz transform and $g_1$ is given by

$$\hat{g}_1(t,\xi) = \int e^{-it\phi(\xi,\eta)} a_0(\xi,\eta)\widehat{\mathcal{R}f}(t,\xi-\eta)\widehat{\mathcal{R}f}(t,\eta)d\eta.$$

Then

$$\begin{aligned}\|\langle x\rangle g\|_{2+\epsilon_1} &\lesssim \||\nabla|^{-1}g_1\|_{2+\epsilon_1} + \|\langle x\rangle g_1\|_{2+\epsilon_1}\\ &\lesssim \|\langle x\rangle g_1\|_{2+\tilde{\epsilon}_2} + \|\langle x\rangle g_1\|_{2+\epsilon_1},\end{aligned}$$

where $\tilde{\epsilon}_2 < \epsilon_1$ is slightly smaller than $\epsilon_1$.

Observe that using a few steps of simple integration by parts,

$$\widehat{xg_1}(t,\xi) \sim t\int (\partial_\xi\phi + \partial_\eta\phi)e^{-it\phi}a_0(\xi,\eta)\widehat{\mathcal{R}f}(t,\xi-\eta)\widehat{\mathcal{R}f}(t,\eta)d\eta \tag{4.1}$$

$$+\int e^{-it\phi}(\partial_\xi a_0 + \partial_\eta a_0)\widehat{\mathcal{R}f}(t,\xi-\eta)\widehat{\mathcal{R}f}(t,\eta)d\eta \tag{4.2}$$

$$+\int e^{-it\phi}a_0\widehat{\mathcal{R}f}(t,\xi-\eta)\widehat{|\nabla|^{-1}f}(t,\eta)d\eta. \tag{4.3}$$

$$+\int e^{-it\phi}a_0\widehat{\mathcal{R}f}(t,\xi-\eta)\frac{\eta}{|\eta|}\widehat{xf}(t,\eta)d\eta. \tag{4.4}$$

`Estimate of` (4.1): In the real space, (4.1) is given by the expression

$$te^{-it\langle\nabla\rangle}T_{(\partial_\xi\phi+\partial_\eta\phi)a_0(\xi,\eta)}(\mathcal{R}h,\mathcal{R}h). \tag{4.5}$$

Let $p$ be such that $2\le p\le 2+\epsilon_1$, then recall the parameters $1\ll N_1\ll N'$,

$$\begin{aligned}&\|e^{-it\langle\nabla\rangle}T_{(\partial_\xi\phi+\partial_\eta\phi)a_0(\xi,\eta)}(\mathcal{R}h,\mathcal{R}h)\|_p\\ \lesssim&\||\nabla|^{1-\frac{2}{p}}(T_{(\partial_\xi\phi+\partial_\eta\phi)a_0(\xi,\eta)\langle\xi-\eta\rangle^{-N_1}\langle\eta\rangle^{-N_1}}(\mathcal{R}\langle\nabla\rangle^{N_1}h,\mathcal{R}\langle\nabla\rangle^{N_1}h))\|_2\\ \lesssim&\|\langle\nabla\rangle^{2N_1}h\|_4^2\\ \lesssim&\|\langle\nabla\rangle^{N'}h\|_2^{\frac{2}{3}}\|h\|_8^{\frac{4}{3}}\\ \lesssim&\frac{1}{\langle t\rangle}\|h\|_X^2.\end{aligned}$$

Clearly this is enough for us.

The estimates of (4.2) and (4.3) are fairly easy so we skip them. For (4.4), note that the frequency variables $(\xi-\eta,\eta)$ are all localized to $|(\xi-\eta,\eta)|\lesssim\langle t\rangle^{\delta_1}$. Hence by Bernstein and Lemma 2.2,

$$\begin{aligned}\|(4.4)\|_p \lesssim&\|P_{\lesssim\langle t\rangle^{\delta_1}}e^{-it\langle\nabla\rangle}T_{a_0(\xi,\eta)}(\mathcal{R}h,\mathcal{R}P_{\le\langle t\rangle^{\delta_1}}e^{it\langle\nabla\rangle}(xf))\|_p\\ \lesssim&\langle t\rangle^{C\epsilon_1+C\delta_1}\|h\|_{\infty-}\|xf\|_{2+\epsilon_1}\\ \lesssim&\|h\|_X^2.\end{aligned}$$

Lemma is now proved. □

Consider next the high frequency piece:

$$\hat{g}_2(t,\xi)=\frac{\xi}{|\xi|}\int e^{-it\phi}a_2(\xi,\eta)\widehat{\mathcal{R}f}(t,\xi-\eta)\widehat{\mathcal{R}f}(t,\eta)d\eta,$$

where

$$a_2(\xi,\eta)=\tilde{a}(\xi,\eta)\cdot\chi_{|\xi-\eta|>\langle t\rangle^{\delta_1}}.$$

We only need to discuss this case in view of the symmetry between $\eta$ and $\xi-\eta$. The other case $|\eta|\gtrsim\langle t\rangle^{\delta_1}$ is omitted.

Then

**Lemma 4.5.**

$$\|\langle x\rangle g_2\|_{2+\epsilon_1}\lesssim\|h\|_X^2.$$

*Proof.* By the discussion in the beginning part of the proof of Lemma 4.4, we see that we only need to bound for $2\le p\le 2+\epsilon_1$, the quantity

$$\|\langle x\rangle\tilde{g}_2\|_p,$$

where

$$\widehat{\tilde{g}_2}(t,\xi)=\int e^{-it\phi(\xi,\eta)}a_2(\xi,\eta)\widehat{\mathcal{R}f}(t,\xi-\eta)\widehat{\mathcal{R}f}(t,\eta)d\eta.$$

Observe

$$\partial_\xi\widehat{\tilde{g}_2}(t,\xi)\sim\int e^{-it\phi}(-it\partial_\xi\phi\cdot a_2+\partial_\xi a_2)\cdot\widehat{\mathcal{R}f}(t,\xi-\eta)\widehat{\mathcal{R}f}(t,\eta)d\eta \tag{4.6}$$

$$+\int e^{-it\phi}a_2\partial_\xi(\widehat{\mathcal{R}f}(t,\xi-\eta))\widehat{\mathcal{R}f}(t,\eta)d\eta. \tag{4.7}$$

The estimate of (4.6) is rather easy:

$$\begin{aligned}\|\mathcal{F}^{-1}((4.6))\|_p&\lesssim\langle t\rangle\|\langle\nabla\rangle^{N_1}h\|_2\cdot\|\langle\nabla\rangle^{N_1}P_{>\langle t\rangle^{\delta_1}}h\|_2\\&\lesssim\langle t\rangle^{1-\frac{N}{2}\delta_1}\|h\|_X^2\\&\lesssim\|h\|_X^2,\end{aligned}$$

where we used the fact $N\gg 1/\delta_1$.

It remains to estimate (4.7). We decompose it further as

$$(4.7)=\int e^{-it\phi}a_2\chi_{|\eta|\ge|\xi-\eta|}\partial_\xi(\widehat{\mathcal{R}f}(t,\xi-\eta))\widehat{\mathcal{R}f}(t,\eta)d\eta. \tag{4.8}$$

$$+\int e^{-it\phi}a_2\chi_{|\eta|<|\xi-\eta|}\partial_\xi(\widehat{\mathcal{R}f}(t,\xi-\eta))\widehat{\mathcal{R}f}(t,\eta)d\eta. \tag{4.9}$$

Here $\chi_{|\eta|\ge|\xi-\eta|}=1-\chi_{|\eta|<|\xi-\eta|}$ and

$$\chi_{|\eta|<|\xi-\eta|}=\phi_{<1}\left(\frac{\eta}{|\xi-\eta|}\right).$$

In particular $\chi_{|\eta|<|\xi-\eta|}$ is smooth in $\eta$ since the symbol $a_2(\xi,\eta)$ is localized to $|\xi-\eta|\gtrsim 1$.

By frequency localization and Lemma 2.2,

$$\begin{aligned}
&\|\mathcal{F}^{-1}((4.8))\|_p\\
\lesssim&\|T_{\langle\xi\rangle^{N_1}a_2\cdot\chi_{|\eta|\ge|\xi-\eta|}\langle\eta\rangle^{-10N_1}\langle\xi-\eta\rangle^{N_1}}\Big(|\nabla|^{-1}P_{\gtrsim 1}h,\langle\nabla\rangle^{10N_1}P_{\gtrsim\langle t\rangle^{\delta_1}}\mathcal{R}h\Big)\|_2\\
&+\|T_{\langle\xi\rangle^{N_1}a_2\cdot\chi_{|\eta|\ge|\xi-\eta|}\langle\eta\rangle^{-10N_1}\langle\xi-\eta\rangle^{N_1}}\Big(\langle\nabla\rangle^{-N_1}P_{\ge 1}e^{it\langle\nabla\rangle}\mathcal{R}(xf),\langle\nabla\rangle^{10N_1}P_{\gtrsim\langle t\rangle^{\delta_1}}\mathcal{R}h\Big)\|_2\\
\lesssim&\|h\|_X^2+\langle t\rangle^{-100}\|P_{\ge1}\langle\nabla\rangle^{-N_1}e^{it\langle\nabla\rangle}(xf)\|_{2+\epsilon_1}\|h\|_X\\
\lesssim&\|h\|_X^2.
\end{aligned}$$

For (4.9) we note that

$$\partial_\xi\left(\widehat{\mathcal{R}f}(t,\xi-\eta)\right)=-\partial_\eta\left(\widehat{\mathcal{R}f}(t,\xi-\eta)\right).$$

Integrating by parts in $\eta$ gives us

$$(4.9)=\int\partial_\eta\Big(e^{-it\phi}a_2\chi_{|\eta|<|\xi-\eta|}\Big)\widehat{\mathcal{R}f}(t,\xi-\eta)\widehat{\mathcal{R}f}(t,\eta)d\eta \tag{4.10}$$

$$+\int e^{-it\phi}a_2\chi_{|\eta|<|\xi-\eta|}\widehat{\mathcal{R}f}(t,\xi-\eta)\widehat{|\nabla|^{-1}f}(t,\eta)d\eta \tag{4.11}$$

$$+\int e^{-it\phi}a_2\chi_{|\eta|<|\xi-\eta|}\widehat{\mathcal{R}f}(t,\xi-\eta)\frac{\eta}{|\eta|}\widehat{xf}(t,\eta)d\eta. \tag{4.12}$$

The term (4.10) can be estimated in the same way as (4.6) and we skip it. The estimate of the term (4.11) is also easy using the boundedness of $\|xf\|_{2+\epsilon_1}$ and the localization $|\eta|\lesssim|\xi-\eta|$. The term (4.12) can also be estimated in a similar fashion as (4.8).

Therefore we have proved

$$\|\langle x\rangle\tilde{g}_2\|_p\lesssim\|h\|_X^2.$$

□

## 5. Control of cubic interactions: high frequency piece

In this section we control the cubic part of the solution which takes the form (see (3.9)):

$$\hat{f}_{cubic}(t,\xi)=\int_0^t\int e^{-is\phi(\xi,\eta,\sigma)}m_1(\xi,\eta,\sigma)\hat{f}(s,\xi-\eta)\hat{f}(s,\eta-\sigma)\hat{f}(s,\sigma)d\eta d\sigma ds,$$

where

$$m_1(\xi,\eta,\sigma)=a(\xi,\eta)b(\eta,\sigma)\frac{\xi}{|\xi|}\frac{\xi-\eta}{|\xi-\eta|}\frac{\eta}{|\eta|}\frac{\eta}{|\eta|}\frac{\eta-\sigma}{|\eta-\sigma|}\frac{\sigma}{|\sigma|}.$$

Here $a,b$ are $C^\infty$ functions with the point-wise bounds:

$$\begin{aligned}
&|\partial_\xi^\alpha\partial_\eta^\beta a(\xi,\eta)|\lesssim_{\alpha,\beta}\langle|\xi|+|\eta|\rangle^{1+|\alpha|+|\beta|},\\
&|\partial_\eta^\alpha\partial_\sigma^\beta b(\eta,\sigma)|\lesssim_{\alpha,\beta}\langle\eta\rangle.
\end{aligned} \tag{5.1}$$

We write $f_{cubic} = \mathcal{R}f_3$ ($\mathcal{R}$ is the Riesz transform), where

$$\begin{aligned}
&\hat{f}_3(t,\xi) \\
&= \int_0^t \int e^{-is\phi(\xi,\eta,\sigma)} m(\xi,\eta,\sigma)\widehat{\mathcal{R}f}(s,\xi-\eta)\frac{\eta}{|\eta|}\frac{\eta}{|\eta|}\Big(\widehat{\mathcal{R}f}(\eta-\sigma)\widehat{\mathcal{R}f}(\sigma)\Big)d\eta d\sigma ds, \quad (5.2)
\end{aligned}$$

and $m(\xi,\eta,\sigma)$ is separable in the sense that

$$m(\xi,\eta,\sigma) = a(\xi,\eta)b(\eta,\sigma). \quad (5.3)$$

Note that in the expression (5.2), the frequency variables $(\xi,\eta,\sigma)$ are vectors and the expression $\frac{\eta}{|\eta|}\frac{\eta}{|\eta|}$ is actually a 2 by 2 matrix, i.e.:

$$\Big(\frac{\eta}{|\eta|}\frac{\eta}{|\eta|}\Big)_{jk} = \frac{\eta_j\eta_k}{|\eta|^2}.$$

This is just the symbol of the usual Riesz-type transforms and their precise form will not be important in our analysis. Therefore to simplify the notations we shall simply write

$$\frac{\eta}{|\eta|}\frac{\eta}{|\eta|} = \frac{\eta}{|\eta|}$$

and assume everything is scalar-valued. Then (5.2) takes the form

$$\begin{aligned}
&\hat{f}_3(t,\xi) \\
&= \int_0^t \int e^{-is\phi(\xi,\eta,\sigma)} m(\xi,\eta,\sigma)\widehat{\mathcal{R}f}(s,\xi-\eta)\frac{\eta}{|\eta|}\Big(\widehat{\mathcal{R}f}(\eta-\sigma)\widehat{\mathcal{R}f}(\sigma)\Big)d\eta d\sigma ds. \quad (5.4)
\end{aligned}$$

Differentiating $\hat{f}_3(t,\xi)$ in $\xi$ gives

$$\begin{aligned}
&\partial_\xi \hat{f}_3(t,\xi) \\
&= -i\int_0^t \int s(\partial_\xi\phi)e^{-is\phi}\cdot m\cdot \widehat{\mathcal{R}f}(s,\xi-\eta)\frac{\eta}{|\eta|}\Big(\widehat{\mathcal{R}f}(s,\eta-\sigma)\widehat{\mathcal{R}f}(\sigma)\Big)d\sigma d\eta ds \quad (5.5)\\
&\quad + \int_0^t \int e^{-is\phi}\partial_\xi m\cdot \widehat{\mathcal{R}f}(s,\xi-\eta)\frac{\eta}{|\eta|}\Big(\widehat{\mathcal{R}f}(s,\eta-\sigma)\widehat{\mathcal{R}f}(\sigma)\Big)d\sigma d\eta ds \quad (5.6)\\
&\quad + \int_0^t \int e^{-is\phi} m\cdot \partial_\xi(\widehat{\mathcal{R}f}(s,\xi-\eta))\frac{\eta}{|\eta|}\Big(\widehat{\mathcal{R}f}(s,\eta-\sigma)\widehat{\mathcal{R}f}(\sigma)\Big)d\sigma d\eta ds. \quad (5.7)
\end{aligned}$$

We have the following

**Proposition 5.1** (Reduction to low frequency estimates)**.** *We have*

$$\|e^{it\langle\nabla\rangle} f_{cubic}\|_{X_1} \lesssim \|h\|_X^3 + \|f_{low}\|_{L_t^\infty L_x^{2-\epsilon_1}([1,\infty)\times\mathbb{R}^2)} \quad (5.8)$$

*where*

$$\begin{aligned}
&\hat{f}_{low}(t,\xi) \\
&= \int_1^t \int s(\partial_\xi\phi)e^{-is\phi}\cdot m_{low}\cdot \widehat{\mathcal{R}f}(s,\xi-\eta)\frac{\eta}{|\eta|}\Big(\widehat{\mathcal{R}f}(s,\eta-\sigma)\widehat{\mathcal{R}f}(\sigma)\Big)d\sigma d\eta ds \quad (5.9)
\end{aligned}$$

*and*

$$m_{low}(\xi,\eta,\sigma) = a_{low}(\xi,\eta)b_{low}(\eta,\sigma),$$

*with*

$$a_{low}(\xi,\eta)=a(\xi,\eta)\chi_{|\xi-\eta|\lesssim\langle s\rangle^{\delta_1}}\chi_{|\eta|\lesssim\langle s\rangle^{\delta_1}},$$
$$b_{low}(\eta,\sigma)=b(\eta,\sigma)\chi_{|\eta-\sigma|\lesssim\langle s\rangle^{\delta_1}}\chi_{|\sigma|\lesssim\langle s\rangle^{\delta_1}}.$$

*Here $a$, $b$ are $C^\infty$ functions satisfying the point-wise bounds:*

$$|\partial_\xi^\alpha\partial_\eta^\beta a(\xi,\eta)|\lesssim_{\alpha,\beta}(1+|\xi|+|\eta|)^{1+|\alpha|+|\beta|},\qquad \forall\ \xi,\eta\in\mathbb{R}^2;$$
$$|\partial_\eta^\alpha\partial_\sigma^\beta b(\eta,\sigma)|\lesssim_{\alpha,\beta}\langle\eta\rangle,\qquad \forall\ \eta,\sigma\in\mathbb{R}^2.$$

*Proof.* By the definition of the $X_1$-norm and the fact $f_{cubic}=\mathcal{R}f_3$, we have

$$\|e^{it\langle\nabla\rangle}f_{cubic}(t)\|_{X_1}=\|\langle t\rangle|\nabla|^{\frac12}\langle\nabla\rangle e^{it\langle\nabla\rangle}\mathcal{R}f_3(t)\|_{L_t^\infty L_x^\infty}+\|\langle t\rangle^{1-\frac{2}{q}}e^{it\langle\nabla\rangle}\mathcal{R}f_3(t)\|_{L_t^\infty L_x^q}$$
$$+\|f_3(t)\|_{L_t^\infty H^{N'}}+\|\langle x\rangle\mathcal{R}f_3(t)\|_{L_t^\infty L_x^{2+\epsilon_1}}.$$

We start with the $H^{N'}$-estimate. Write $f_3$ as

$$f_3(t)=\int_0^t e^{-is\langle\nabla\rangle}T_{a(\xi,\eta)}\Big(\mathcal{R}h,\mathcal{R}T_{b(\eta,\sigma)}(\mathcal{R}h,\mathcal{R}h)\Big)ds.$$

Then clearly

$$\|f_3\|_{H^{N'}}\lesssim\int_0^t\Big\|T_{a(\xi,\eta)}\Big(\mathcal{R}h,\mathcal{R}T_{b(\eta,\sigma)}(\mathcal{R}h,\mathcal{R}h)\Big)\Big\|_{H^{N'}}ds$$
$$\lesssim\int_0^t\|\langle\nabla\rangle^{2N'}h\|_6^3ds.$$

Since $N'\ll N$,

$$\|\langle\nabla\rangle^{2N'}h(s)\|_6\lesssim\|h(s)\|_{18}^{\frac34}\|\langle\nabla\rangle^N h(s)\|_2^{\frac14}$$
$$\lesssim\langle s\rangle^{-\frac23+C\delta_1}\|h\|_X.$$

we get

$$\|f_3\|_{H^{N'}}\lesssim\int_0^t\langle s\rangle^{-2+C\delta_1}ds\|h\|_X^3$$
$$\lesssim\|h\|_X^3$$

which is acceptable.

By a similar estimate, we also get

$$\|\langle\nabla\rangle^{N'}f_3(t)\|_{L_t^\infty L_x^{2-\epsilon_1}}\lesssim\|h\|_X^3;\tag{5.10}$$

and

$$\|\langle\nabla\rangle^{N'}\Big(\mathcal{F}^{-1}((5.6))\Big)\|_{L_t^\infty L_x^{2-\epsilon_1}}\lesssim\|h\|_X^3.\tag{5.11}$$

These two estimates will be used below.

Now we turn to other norms. Observe that

$$\||\nabla|^{\frac12}\langle\nabla\rangle e^{it\langle\nabla\rangle}\mathcal{R}f_3(t)\|_\infty\lesssim\sum_{M<1}M^{\frac12}\|P_Me^{it\langle\nabla\rangle}f_3(t)\|_\infty+\sum_{M\ge1}M^{\frac32}\|P_Me^{it\langle\nabla\rangle}f_3(t)\|_\infty$$
$$\lesssim\langle t\rangle^{-1}\|\langle\nabla\rangle^9f_3(t)\|_1.$$

Also obviously

$$\begin{aligned}\|e^{it\langle\nabla\rangle}\mathcal{R}f_3(t)\|_q &\lesssim \langle t\rangle^{\frac{2}{q}-1}\|\langle\nabla\rangle^5 f_3(t)\|_{q/(q-1)}\\ &\lesssim \langle t\rangle^{\frac{2}{q}-1}\|\langle\nabla\rangle^9 f_3(t)\|_1.\end{aligned}$$

On the other hand

$$\begin{aligned}\|\langle x\rangle\mathcal{R}f_3(t)\|_{2+\epsilon_1} &\lesssim \||\nabla|^{-1}f_3\|_{2+\epsilon_1}+\|\langle x\rangle f_3(t)\|_{2+\epsilon_1}\\ &\lesssim \|f_3(t)\|_1+\|\langle x\rangle f_3(t)\|_{2+\epsilon_1}\\ &\lesssim \|\langle\nabla\rangle^9 f_3(t)\|_1+\|\langle x\rangle f_3(t)\|_{2+\epsilon_1}.\end{aligned}$$

Denote the RHS of (5.8) as $Y$. Then we are reduced to showing

$$\|\langle\nabla\rangle^9 f_3(t)\|_{L_t^\infty L_x^1}+\|\langle x\rangle f_3(t)\|_{L_t^\infty L_x^{2+\epsilon_1}}\lesssim Y.$$

Since

$$\begin{aligned}\|\langle\nabla\rangle^9 f_3\|_1 &\lesssim \|\langle x\rangle\big(\langle\nabla\rangle^9 f_3\big)\|_{2-\epsilon_1}\\ &\lesssim \|\langle\nabla\rangle^9 f_3\|_{2-\epsilon_1}+\|\langle\nabla\rangle^9\big(xf_3\big)\|_{2-\epsilon_1},\end{aligned}$$

also

$$\|xf_3(t)\|_{2+\epsilon_1}\lesssim\|\langle\nabla\rangle(xf_3(t))\|_{2-\epsilon_1},$$

it then suffices for us to show

$$\|\langle\nabla\rangle^9 f_3(t)\|_{L_t^\infty L_x^{2-\epsilon_1}}+\|\langle\nabla\rangle^9\big(xf_3(t)\big)\|_{L_t^\infty L_x^{2-\epsilon_1}}\lesssim Y.$$

By (5.10), we only need to show

$$\|\langle\nabla\rangle^9\big(xf_3(t)\big)\|_{L_t^\infty L_x^{2-\epsilon_1}}\lesssim Y.$$

By (5.11), we only need to prove

$$\|\langle\nabla\rangle^9\mathcal{F}^{-1}((5.5))\|_{L_t^\infty L_x^{2-\epsilon_1}}+\|\langle\nabla\rangle^9\mathcal{F}^{-1}((5.7))\|_{L_t^\infty L_x^{2-\epsilon_1}}\lesssim Y.$$

We first deal with (5.5). Write

$$\begin{aligned}&\mathcal{F}^{-1}((5.5))\\ =&\int_0^t se^{-is\langle\nabla\rangle}T_{\partial_\xi\phi a(\xi,\eta)}\Big(\mathcal{R}h,\mathcal{R}T_{b(\eta,\sigma)}(\mathcal{R}h,\mathcal{R}h)\Big)ds\\ =&\int_0^t se^{-is\langle\nabla\rangle}T_{\partial_\xi\phi a(\xi,\eta)}\Big(\mathcal{R}P_{<\langle s\rangle^{\delta_1}}h,\mathcal{R}T_{b(\eta,\sigma)}(\mathcal{R}P_{<\langle s\rangle^{\delta_1}}h,\mathcal{R}P_{<\langle s\rangle^{\delta_1}}h)\Big)ds \qquad (5.12)\\ &+O\left(\int_0^t se^{-is\langle\nabla\rangle}T_{\partial_\xi\phi\cdot a(\xi,\eta)}\Big(\mathcal{R}P_{\ge\langle s\rangle^{\delta_1}}h,\mathcal{R}T_{b(\eta,\sigma)}(\mathcal{R}h,\mathcal{R}h)\Big)ds\right) \qquad (5.13)\\ &+O\left(\int_0^t se^{-is\langle\nabla\rangle}T_{\partial_\xi\phi\cdot a(\xi,\eta)}\Big(\mathcal{R}h,\mathcal{R}T_{b(\eta,\sigma)}(\mathcal{R}P_{\ge\langle s\rangle^{\delta_1}}h,\mathcal{R}h)\Big)ds\right) \qquad (5.14)\\ &+O\left(\int_0^t se^{-is\langle\nabla\rangle}T_{\partial_\xi\phi\cdot a(\xi,\eta)}\Big(\mathcal{R}h,\mathcal{R}T_{b(\eta,\sigma)}(\mathcal{R}h,\mathcal{R}P_{\ge\langle s\rangle^{\delta_1}}h)\Big)ds\right). \qquad (5.15)\end{aligned}$$

Here in (5.13)–(5.15) there is at least one high frequency cut-off $P_{>\langle s\rangle^{\delta_1}}$ on the function $h$. Consider (5.13). By Lemma 2.2, we have

$$\begin{aligned}\|\langle\nabla\rangle^9(5.13)\|_{2-\epsilon_1} &\lesssim \int_0^t \langle s\rangle^2 \|\langle\nabla\rangle^{13} T_{\partial_\xi\phi a(\xi,\eta)}(\mathcal{R}P_{\geq\langle s\rangle^{\delta_1}}h, \mathcal{R}T_{b(\eta,\sigma)}(\mathcal{R}h,\mathcal{R}h))\|_{2-\epsilon_1}ds \\ &\lesssim \int_0^t \langle s\rangle^{2+C\delta_1}\|\langle\nabla\rangle^{N_1}P_{\geq\langle s\rangle^{\delta_1}}h\|_2 ds\|h\|_X^2 \\ &\lesssim \int_0^t \langle s\rangle^{2+C\delta_1-\frac{N}{2}\delta_1}ds\|h\|_X^3 \\ &\lesssim \|h\|_X^3,\end{aligned}$$

where we used the fact $N \gg 1/\delta_1$.

Similarly

$$\|\langle\nabla\rangle^9(5.14)\|_{2-\epsilon_1} + \|\langle\nabla\rangle^9(5.15)\|_{2-\epsilon_1} \lesssim \|h\|_X^3.$$

Also it is not difficult to check that

$$\begin{aligned}\|\langle\nabla\rangle^9(5.12)\|_{L_t^\infty L_x^{2-\epsilon_1}([0,1]\times\mathbb{R}^2)} &\lesssim \int_0^1 \langle s\rangle^C ds\|h\|_X^3 \\ &\lesssim \|h\|_X^3.\end{aligned}$$

Hence we have proved that

$$\|\langle\nabla\rangle^9\mathcal{F}^{-1}((5.5))\|_{L_t^\infty L_x^{2-\epsilon_1}} \lesssim Y.$$

It remains to estimate (5.7). Actually we shall show the slightly stronger estimate

$$\|\langle\nabla\rangle^9\mathcal{F}^{-1}((5.7))\|_{2-\epsilon_1} \lesssim \|h\|_X^3.$$

We decompose (5.7) as

$$\begin{aligned}&(5.7)\\ &= \int_0^t\int e^{-is\phi}m\partial_\xi(\widehat{\mathcal{R}f}(s,\xi-\eta))\chi_{|\xi-\eta|\leq\langle s\rangle^{\delta_1}}\chi_{|\eta|\leq\langle s\rangle^{\delta_1}}\cdot\frac{\eta}{|\eta|}(\widehat{\mathcal{R}f}(s,\eta-\sigma)\widehat{\mathcal{R}f}(s,\sigma))ds \qquad (5.16)\\ &\quad+\int_0^t\int e^{-is\phi}m\partial_\xi(\widehat{\mathcal{R}f}(s,\xi-\eta))\chi_{|\xi-\eta|\leq\langle s\rangle^{\delta_1}}\chi_{|\eta|>\langle s\rangle^{\delta_1}}\cdot\frac{\eta}{|\eta|}(\widehat{\mathcal{R}f}(s,\eta-\sigma)\widehat{\mathcal{R}f}(s,\sigma))ds \qquad (5.17)\\ &\quad+\int_0^t\int e^{-is\phi}m\partial_\xi(\widehat{\mathcal{R}f}(s,\xi-\eta))\chi_{|\xi-\eta|>\langle s\rangle^{\delta_1}}\chi_{|\eta|\leq|\xi-\eta|}\cdot\frac{\eta}{|\eta|}(\widehat{\mathcal{R}f}(s,\eta-\sigma)\widehat{\mathcal{R}f}(s,\sigma))ds \qquad (5.18)\\ &\quad+\int_0^t\int e^{-is\phi}m\partial_\xi(\widehat{\mathcal{R}f}(s,\xi-\eta))\chi_{|\xi-\eta|>\langle s\rangle^{\delta_1}}\chi_{|\eta|>|\xi-\eta|}\cdot\frac{\eta}{|\eta|}(\widehat{\mathcal{R}f}(s,\eta-\sigma)\widehat{\mathcal{R}f}(s,\sigma))ds. \qquad (5.19)\end{aligned}$$

The estimate of (5.16) is quite straightforward. Since $|\xi-\eta| \le \langle s\rangle^{\delta_1}$, and $|\eta| \le \langle s\rangle^{\delta_1}$, we can insert a fattened cut-off $\chi_{|\xi|\lesssim \langle s\rangle^{\delta_1}}$ in (5.16). Hence

$$\begin{aligned}
&\|\langle\nabla\rangle^9 \mathcal{F}^{-1}((5.16))\|_{2-\epsilon_1}\\
\lesssim&\int_0^t \Big\|\langle\nabla\rangle^9 e^{-is\langle\nabla\rangle} P_{\lesssim\langle s\rangle^{\delta_1}} T_{a(\xi,\eta)}\Big(e^{is\langle\nabla\rangle}P_{\le\langle s\rangle^{\delta_1}}(\mathcal{R}(xf)+|\nabla|^{-1}f), P_{\le\langle s\rangle^{\delta_1}}\mathcal{R}T_{b(\eta,\sigma)}(\mathcal{R}h,\mathcal{R}h)\Big)\Big\|_{2-\epsilon_1} ds\\
\lesssim&\int_0^t \langle s\rangle^{C\delta_1+C\epsilon_1-2} ds \|h\|_X^3\\
\lesssim&\|h\|_X^3.
\end{aligned}$$

For (5.17) we note that $|\eta|\gtrsim|\xi-\eta|$ and the symbol $a(\xi,\eta)$ (together with its sufficiently many derivatives) can be controlled by $\langle\eta\rangle^C$ for some large constant $C$. By Lemma 2.2,

$$\begin{aligned}
&\|\langle\nabla\rangle^9 \mathcal{F}^{-1}((5.17))\|_{2-\epsilon_1}\\
\lesssim&\int_0^t \langle s\rangle \Big\|\langle\nabla\rangle^{20} T_{a(\xi,\eta)\langle\eta\rangle^{-N_1}}\Big(e^{is\langle\nabla\rangle}P_{\le\langle s\rangle^{\delta_1}}(\mathcal{R}(xf)+|\nabla|^{-1}f), \langle\nabla\rangle^{N_1}P_{>\langle s\rangle^{\delta_1}}\mathcal{R}T_{b(\eta,\sigma)}(\mathcal{R}h,\mathcal{R}h)\Big)\Big\|_{2-\epsilon_1} ds\\
\lesssim&\int_0^t \langle s\rangle^{1-\frac{N}{2}\delta_1} ds \|h\|_X^3\\
\lesssim&\|h\|_X^3.
\end{aligned}$$

Similarly for (5.19) we can perform a dyadic summation (in $\eta$) to obtain,

$$\begin{aligned}
&\|\langle\nabla\rangle^9 \mathcal{F}^{-1}((5.19))\|_{2-\epsilon_1}\\
\lesssim&\int_0^t \langle s\rangle \sum_{M\gtrsim\langle s\rangle^{\delta_1}} \Big\|\langle\nabla\rangle^{20} T_{a(\xi,\eta)\langle\eta\rangle^{-N_1}}\Big(e^{is\langle\nabla\rangle}P_{\lesssim M}(\mathcal{R}(xf)+|\nabla|^{-1}f), \langle\nabla\rangle^{N_1}P_{M}\mathcal{R}T_{b(\eta,\sigma)}(\mathcal{R}h,\mathcal{R}h)\Big)\Big\|_{2-\epsilon_1} ds\\
\lesssim&\int_0^t \langle s\rangle \sum_{M\gtrsim\langle s\rangle^{\delta_1}} \langle s\rangle M^{N_1-\frac{N}{2}}\langle s\rangle^{C\delta_1} ds \|h\|_X^3\\
\lesssim&\|h\|_X^3.
\end{aligned}$$

Finally we estimate (5.18). Using

$$\partial_\xi(\widehat{\mathcal{R}f}(s,\xi-\eta)) = -\partial_\eta(\widehat{\mathcal{R}f}(s,\xi-\eta)),$$

we obtain

$$\begin{aligned}
&(5.18)\\
=&\int_0^t\int e^{-is\phi}((-ism\partial_\eta\phi+\partial_\eta m)\chi_{|\xi-\eta|>\langle s\rangle^{\delta_1}}\chi_{|\eta|\le|\xi-\eta|} + m\partial_\eta(\chi_{|\xi-\eta|>\langle s\rangle^{\delta_1}}\chi_{|\eta|\le|\xi-\eta|}))\\
&\quad\cdot\widehat{\mathcal{R}f}(s,\xi-\eta)\cdot\frac{\eta}{|\eta|}(\widehat{\mathcal{R}f}(s,\eta-\sigma)\widehat{\mathcal{R}f}(s,\sigma))d\sigma d\eta ds \qquad (5.20)\\
&+\int_0^t\int e^{-is\phi} m\chi_{|\xi-\eta|>\langle s\rangle^{\delta_1}}\chi_{|\eta|\le|\xi-\eta|}\widehat{\mathcal{R}f}(s,\xi-\eta)\partial_\eta(\frac{\eta}{|\eta|})\widehat{\mathcal{R}f}(s,\eta-\sigma)\widehat{\mathcal{R}f}(s,\sigma)d\sigma d\eta ds \qquad (5.21)\\
&+\int_0^t\int e^{-is\phi} m\chi_{|\xi-\eta|>\langle s\rangle^{\delta_1}}\chi_{|\eta|\le|\xi-\eta|}\widehat{\mathcal{R}f}(s,\xi-\eta)\frac{\eta}{|\eta|}(\partial_\eta(\widehat{\mathcal{R}f}(s,\eta-\sigma))\widehat{\mathcal{R}f}(s,\sigma))d\sigma d\eta ds. \qquad (5.22)
\end{aligned}$$

Clearly

$$\begin{aligned}\|\langle\nabla\rangle^9\mathcal{F}^{-1}((5.20))\|_{2-\epsilon_1} &\lesssim \int_0^t \langle s\rangle^{1+C\delta_1}\|\langle\nabla\rangle^{N_1}P_{>\langle s\rangle^{\delta_1}}h\|_2\|\langle\nabla\rangle^{10}h\|_2^2 ds\\ &\lesssim \int_0^t \langle s\rangle^{1+C\delta_1-\frac{N}{2}\delta_1}ds\|h\|_X^3\\ &\lesssim \|h\|_X^3.\end{aligned}$$

Similarly

$$\|\langle\nabla\rangle^9\mathcal{F}^{-1}((5.21))\|_{2-\epsilon_1}+\|\langle\nabla\rangle^9\mathcal{F}^{-1}((5.22))\|_{2-\epsilon_1}\lesssim\|h\|_X^3.$$

This ends the proof of the proposition. □

## 6. Control of cubic interactions: low frequency piece

In the previous section, we controlled the high frequency part of the cubic interaction term. The most delicate part of our analysis comes from the low frequency piece.

**Proposition 6.1.** *We have*

$$\|f_{low}\|_{L_t^\infty L_x^{2-\epsilon_1}([1,\infty)\times\mathbb{R}^2)}\lesssim\|h\|_X^3+\|h\|_X^4,$$

*where $f_{low}$ was defined in Proposition 5.1.*

The proof of this proposition will occupy the rest of this section.
**Notation:** We will use the notation $\|h\|_{\infty-}$ to denote $\|h\|_p$ for some $p\gtrsim\frac{1}{\epsilon_1}$. To simplify notations we do not calculate the specific numerical value of $p$ since we only need its order of magnitude. This notation will also be heavily used in the following calculations.

*Proof of Proposition 6.1.* We will discuss different forms of the phase function $\phi(\xi,\eta,\sigma)$.
`Case 1:`

$$\phi(\xi,\eta,\sigma)=\langle\xi\rangle-\langle\xi-\eta\rangle+\langle\eta-\sigma\rangle-\langle\sigma\rangle. \tag{6.1}$$

By Lemma 2.4, we have

$$\partial_\xi\phi=Q_1(\xi,\eta)Q_2(\eta,\sigma)\partial_\sigma\phi,$$

where $Q_1$, $Q_2$ are smooth.
This allows us to write

$$-is\partial_\xi\phi e^{-is\phi}=Q_1(\xi,\eta)Q_2(\eta,\sigma)\partial_\sigma\Big(e^{-is\phi}\Big). \tag{6.2}$$

Plugging (6.2) into (5.9) and integrating by parts in $\sigma$, we then obtain

$$\begin{aligned}
&\hat{f}_{low}(t,\xi)\\
=&-\int_1^t\int Q_1(\xi,\eta)a_{low}(\xi,\eta)\partial_\sigma(Q_2(\eta,\sigma)b_{low}(\eta,\sigma))e^{-is\phi}\\
&\quad\cdot\widehat{\mathcal{R}f}(s,\xi-\eta)\frac{\eta}{|\eta|}(\widehat{\mathcal{R}f}(s,\eta-\sigma)\widehat{\mathcal{R}f}(s,\sigma))d\sigma d\eta ds \qquad (6.3)\\
&-\int_1^t\int Q_1(\xi,\eta)Q_2(\eta,\sigma)e^{-is\phi}a_{low}(\xi,\eta)b_{low}(\eta,\sigma)\widehat{\mathcal{R}f}(s,\xi-\eta)\\
&\quad\cdot\frac{\eta}{|\eta|}\Big(\partial_\sigma(\widehat{\mathcal{R}f}(s,\sigma))\widehat{\mathcal{R}f}(s,\eta-\sigma)\Big)d\sigma d\eta ds. \qquad (6.4)\\
&-\int_1^t\int Q_1(\xi,\eta)Q_2(\eta,\sigma)e^{-is\phi}a_{low}(\xi,\eta)b_{low}(\eta,\sigma)\widehat{\mathcal{R}f}(s,\xi-\eta)\\
&\quad\cdot\frac{\eta}{|\eta|}\Big(\widehat{\mathcal{R}f}(s,\sigma)\partial_\sigma(\widehat{\mathcal{R}f}(s,\eta-\sigma))\Big)d\sigma d\eta ds. \qquad (6.5)
\end{aligned}$$

By Lemma 2.2, we have

$$\begin{aligned}
&\|\mathcal{F}^{-1}((6.3))\|_{2-\epsilon_1}\\
\lesssim&\int_1^t s^{C\epsilon_1}\|T_{Q_1(\xi,\eta)a_{low}(\xi,\eta)}\Big(\mathcal{R}h,\mathcal{R}T_{\partial_\sigma(Q_2(\eta,\sigma)b_{low}(\eta,\sigma))}(\mathcal{R}h,\mathcal{R}h)\Big)\|_{2-\epsilon_1}ds\\
\lesssim&\int_1^t s^{C\epsilon_1+C\delta_1}\|h\|_2\|h\|_{\infty-}^2ds\\
\lesssim&\int_1^t s^{C\epsilon_1+C\delta_1-2}ds\|h\|_X^3\\
\lesssim&\|h\|_X^3.
\end{aligned}$$

Clearly this is acceptable for us.

For (6.4), note that by Lemma 2.2 and Bernstein,

$$\begin{aligned}
&\|e^{is\langle\nabla\rangle}\mathcal{F}^{-1}\Big(\partial_\sigma(\widehat{\mathcal{R}f}(s,\sigma))\chi_{|\sigma|\lesssim\langle s\rangle^{\delta_1}}\Big)\|_{2+100\epsilon_1}\\
\lesssim&\langle s\rangle^{C\delta_1}(\||\nabla|^{-1}f\|_{2+100\epsilon_1}+\|P_{\lesssim\langle s\rangle^{\delta_1}}(xf)\|_{2+100\epsilon_1})\\
\lesssim&\langle s\rangle^{C\delta_1+C\epsilon_1}\|\langle x\rangle f\|_{2+\epsilon_1}.
\end{aligned}$$

Hence

$$\begin{aligned}
&\|\mathcal{F}^{-1}((6.4))\|_{2-\epsilon_1}\\
\lesssim&\int_1^t\langle s\rangle^{C\delta_1+C\epsilon_1}\|h\|_{\infty-}^2\|\langle x\rangle f\|_{2+\epsilon_1}ds\\
\lesssim&\int_1^t\langle s\rangle^{C\delta_1+C\epsilon_1-2}ds\|h\|_X^3\\
\lesssim&\|h\|_X^3.
\end{aligned}$$

By a similar estimate, we also have

$$\|\mathcal{F}^{-1}((6.5))\|_{2-\epsilon_1}\lesssim\|h\|_X^3.$$

This ends the estimate of phase (6.1).

`Case 2`:

$$\phi(\xi,\eta,\sigma) = \langle\xi\rangle - \langle\xi-\eta\rangle - \langle\eta-\sigma\rangle + \langle\sigma\rangle.$$

This is almost the same as the case (6.1) after the change of variable $\sigma \to \eta - \sigma$. We omit the estimates.

`Case 3`:

$$\phi(\xi,\eta,\sigma) = \langle\xi\rangle + \langle\xi-\eta\rangle - \langle\eta-\sigma\rangle - \langle\sigma\rangle. \tag{6.6}$$

For this we will have to exploit some delicate cancelations of the phases. We discuss several subcases.

`Subcase 3a:` $|\eta| \le \langle s\rangle^{-\delta_2}$ and $|\xi| \le \langle s\rangle^{-\delta_2/N_1}$. In the real space, this subcase corresponds to putting the LP projections $P_{\le\langle s\rangle^{-\delta_2}}$ and $P_{\le\langle s\rangle^{-\delta_2/N_1}}$ into corresponding places of the nonlinearity.

Note that

$$\partial_\xi\phi = \frac{\xi}{\langle\xi\rangle} + \frac{\xi-\eta}{\langle\xi-\eta\rangle}.$$

Since $\xi$ and $\xi-\eta$ are both localized to low frequencies, we gain one derivative by using the above identity. In this subcase we have

$$\begin{aligned}
&\|\mathcal{F}^{-1}((5.9))\|_{2-\epsilon_1} \\
\lesssim &\int_1^t s^{1+C\epsilon_1}\|\nabla P_{\lesssim\langle s\rangle^{-\delta_2/N_1}} T_{a_{low}(\xi,\eta)}\Big(\mathcal{R}h, \mathcal{R}T_{b_{low}(\eta,\sigma)}(\mathcal{R}h,\mathcal{R}h)\Big)\|_{2-\epsilon_1} ds \\
&\qquad + \int_1^t s^{1+C\epsilon_1}\|T_{a_{low}(\xi,\eta)}\Big(\nabla P_{\lesssim\langle s\rangle^{-\delta_2/N_1}}\mathcal{R}h, \mathcal{R}T_{b_{low}(\eta,\sigma)}(\mathcal{R}h,\mathcal{R}h)\Big)\|_{2-\epsilon_1} ds \\
\lesssim &\int_1^t s^{1+C\epsilon_1-\frac{\delta_2}{N_1}+C\delta_1} s^{-2} ds \|h\|_X^3 \\
\lesssim &\|h\|_X^3.
\end{aligned}$$

Here we used the fact $\delta_2/N_1 \gg \max\{\epsilon_1,\delta_1\}$.

`Subcase 3b:` $|\eta| \le \langle s\rangle^{-\delta_2}$, $|\xi| > \langle s\rangle^{-\delta_2/N_1}$ and $|\sigma| < \langle s\rangle^{-\delta_2/10}$. In this case we write

$$e^{-is\phi} = e^{-is(\langle\xi\rangle+\langle\xi-\eta\rangle-2)}e^{-is(2-\langle\eta-\sigma\rangle-\langle\sigma\rangle)}.$$

Note that

$$|\langle\xi\rangle + \langle\xi-\eta\rangle - 2| \gtrsim \langle s\rangle^{-\delta_2/N_1}, \tag{6.7}$$

$$\langle\eta-\sigma\rangle + \langle\sigma\rangle - 2 = b_2(\eta,\sigma)\cdot(\eta-\sigma) + b_3(\eta,\sigma)\cdot\sigma, \tag{6.8}$$

where $b_2$, $b_3$ are smooth.

We do an integration by parts using only *part of the phase*. Namely using the identity

$$-ie^{-is(\langle\xi\rangle+\langle\xi-\eta\rangle-2)} = \frac{1}{\langle\xi\rangle+\langle\xi-\eta\rangle-2}\partial_s(e^{-is(\langle\xi\rangle+\langle\xi-\eta\rangle-2)}).$$

and integrating by parts in the time variable $s$, we obtain

$$
\begin{aligned}
&\hat{f}_{low}(t,\xi)\\
=&\int_1^t\int s(\partial_\xi\phi)m_{low}\cdot\chi_{|\eta|\le\langle s\rangle^{-\delta_2}}\chi_{|\xi|>\langle s\rangle^{-\delta_2/N_1}}\chi_{|\sigma|<\langle s\rangle^{-\delta_2/10}}\\
&\qquad\cdot\frac{1}{\langle\xi\rangle+\langle\xi-\eta\rangle-2}\partial_s\Big(e^{-is(\langle\xi\rangle+\langle\xi-\eta\rangle-2)}\Big)e^{-is(2-\langle\eta-\sigma\rangle-\langle\sigma\rangle)}\\
&\qquad\cdot\widehat{\mathcal{R}f}(s,\xi-\eta)\frac{\eta}{|\eta|}\Big(\widehat{\mathcal{R}f}(s,\eta-\sigma)\widehat{\mathcal{R}f}(s,\sigma)\Big)d\sigma d\eta ds\\
=&\int s(\partial_\xi\phi)m_{low}\cdot\chi_{|\eta|\le\langle s\rangle^{-\delta_2}}\chi_{|\xi|>\langle s\rangle^{-\delta_2/N_1}}\chi_{|\sigma|<\langle s\rangle^{-\delta_2/10}}\\
&\qquad\cdot\frac{e^{-is\phi}}{\langle\xi\rangle+\langle\xi-\eta\rangle-2}\cdot\widehat{\mathcal{R}f}(s,\xi-\eta)\frac{\eta}{|\eta|}\Big(\widehat{\mathcal{R}f}(s,\eta-\sigma)\widehat{\mathcal{R}f}(s,\sigma)\Big)d\sigma d\eta\Big|_{s=1}^{t} \qquad (6.9)\\
&-\int_1^t\int(\partial_\xi\phi)\partial_s\Big(sm_{low}\cdot\chi_{|\eta|\le\langle s\rangle^{-\delta_2}}\chi_{|\xi|>\langle s\rangle^{-\delta_2/N_1}}\chi_{|\sigma|<\langle s\rangle^{-\delta_2/10}}\Big)\\
&\qquad\cdot\frac{e^{-is\phi}}{\langle\xi\rangle+\langle\xi-\eta\rangle-2}\cdot\widehat{\mathcal{R}f}(s,\xi-\eta)\frac{\eta}{|\eta|}\Big(\widehat{\mathcal{R}f}(s,\eta-\sigma)\widehat{\mathcal{R}f}(s,\sigma)\Big)d\sigma d\eta ds \qquad (6.10)\\
&-\int_1^t\int(\partial_\xi\phi)\Big(sm_{low}\cdot\chi_{|\eta|\le\langle s\rangle^{-\delta_2}}\chi_{|\xi|>\langle s\rangle^{-\delta_2/N_1}}\chi_{|\sigma|<\langle s\rangle^{-\delta_2/10}}\Big)e^{-is\phi}\\
&\qquad\cdot\frac{-i(2-\langle\eta-\sigma\rangle-\langle\sigma\rangle)}{\langle\xi\rangle+\langle\xi-\eta\rangle-2}\cdot\widehat{\mathcal{R}f}(s,\xi-\eta)\frac{\eta}{|\eta|}\Big(\widehat{\mathcal{R}f}(s,\eta-\sigma)\widehat{\mathcal{R}f}(s,\sigma)\Big)d\sigma d\eta ds \qquad (6.11)\\
&-\int_1^t\int(\partial_\xi\phi)\Big(sm_{low}\cdot\chi_{|\eta|\le\langle s\rangle^{-\delta_2}}\chi_{|\xi|>\langle s\rangle^{-\delta_2/N_1}}\chi_{|\sigma|<\langle s\rangle^{-\delta_2/10}}\Big)e^{-is\phi}\\
&\qquad\cdot\frac{1}{\langle\xi\rangle+\langle\xi-\eta\rangle-2}\cdot\partial_s(\widehat{\mathcal{R}f}(s,\xi-\eta))\frac{\eta}{|\eta|}\Big(\widehat{\mathcal{R}f}(s,\eta-\sigma)\widehat{\mathcal{R}f}(s,\sigma)\Big)d\sigma d\eta ds \qquad (6.12)\\
&-\int_1^t\int(\partial_\xi\phi)\Big(sm_{low}\cdot\chi_{|\eta|\le\langle s\rangle^{-\delta_2}}\chi_{|\xi|>\langle s\rangle^{-\delta_2/N_1}}\chi_{|\sigma|<\langle s\rangle^{-\delta_2/10}}\Big)e^{-is\phi}\\
&\qquad\cdot\frac{1}{\langle\xi\rangle+\langle\xi-\eta\rangle-2}\cdot(\widehat{\mathcal{R}f}(s,\xi-\eta))\frac{\eta}{|\eta|}\Big(\partial_s(\widehat{\mathcal{R}f}(s,\eta-\sigma)\widehat{\mathcal{R}f}(s,\sigma))\Big)d\sigma d\eta ds \qquad (6.13)
\end{aligned}
$$

For (6.9), we have by (6.7),

$$
\begin{aligned}
&\|\mathcal{F}^{-1}((6.9))\|_{2-\epsilon_1}\\
\lesssim&\|h\|_X^3+t^{1+C\epsilon_1}\Big\|T_{\frac{a_{low}(\xi,\eta)\partial_\xi\phi}{\langle\xi\rangle+\langle\xi-\eta\rangle-2}}\Big(\mathcal{R}h(t),\mathcal{R}P_{\lesssim\langle t\rangle^{-\delta_2}}T_{b_{low}(\eta,\sigma)}(\mathcal{R}h(t),\mathcal{R}P_{<\langle t\rangle^{-\delta_2/10}}h(t))\Big)\Big\|_{2-\epsilon_1}\\
\lesssim&\|h\|_X^3+t^{1+C\epsilon_1+C\delta_1+C\delta_2}\|h(t)\|_2\|h(t)\|_{\infty-}^2\\
\lesssim&\|h\|_X^3.
\end{aligned}
$$

For (6.10), it is not difficult to check that

$$
\begin{aligned}
&\partial_s\Big(sm_{low}\cdot\chi_{|\eta|\le\langle s\rangle^{-\delta_2}}\chi_{|\xi|>\langle s\rangle^{-\delta_2/N_1}}\chi_{|\sigma|<\langle s\rangle^{-\delta_2/10}}\Big)\\
=&a(\xi,\eta)b(\eta,\sigma)\tilde{\chi}_{|\xi-\eta|\lesssim\langle s\rangle^{\delta_1}}\tilde{\chi}_{|\eta|\lesssim\langle s\rangle^{\delta_1}}\tilde{\chi}_{|\eta-\sigma|\lesssim\langle s\rangle^{\delta_1}}\tilde{\chi}_{|\sigma|\lesssim\langle s\rangle^{\delta_1}}\tilde{\chi}_{|\eta|\lesssim\langle s\rangle^{-\delta_2}}\tilde{\chi}_{|\xi|\gtrsim\langle s\rangle^{-\delta_2/N_1}}\tilde{\chi}_{|\sigma|\lesssim\langle s\rangle^{-\delta_2/10}},
\end{aligned}
$$

where $\tilde{\chi}$ are some fattened cut-offs. Using again (6.7), we have

$$\begin{aligned}&\|\mathcal{F}^{-1}((6.10))\|_{2-\epsilon_1}\\ \lesssim&\int_1^t s^{C\epsilon_1+C\delta_2+C\delta_1}\|h\|_{\infty-}^2\|h\|_2 ds\\ \lesssim&\|h\|_X^3.\end{aligned}$$

For (6.11), we use (6.8) to gain one more derivative.

$$\begin{aligned}&\|\mathcal{F}^{-1}((6.11))\|_{2-\epsilon_1}\\ \lesssim&\int_1^t s^{1+C\epsilon_1}\left\|T_{\frac{a_{low}(\xi,\eta)}{\langle\xi\rangle+\langle\xi-\eta\rangle-2}}\Big(\mathcal{R}h,\mathcal{R}P_{\le\langle s\rangle^{-\delta_2}}T_{b_2(\eta,\sigma)b_{low}(\eta,\sigma)}(\mathcal{R}\nabla P_{\lesssim\langle s\rangle^{-\delta_2/10}}h,P_{<\langle s\rangle^{-\delta_2/10}}h)\Big)\right\|_{2-\epsilon_1}ds\\ &+\int_1^t s^{1+C\epsilon_1}\left\|T_{\frac{a_{low}(\xi,\eta)}{\langle\xi\rangle+\langle\xi-\eta\rangle-2}}\Big(\mathcal{R}h,\mathcal{R}P_{\le\langle s\rangle^{-\delta_2}}T_{b_3(\eta,\sigma)b_{low}(\eta,\sigma)}(\mathcal{R}h,P_{<\langle s\rangle^{-\delta_2/10}}\nabla h)\Big)\right\|_{2-\epsilon_1}ds\\ \lesssim&\int_1^t s^{1+C\epsilon_1+C\delta_1-\frac{\delta_2}{10}+\frac{20\delta_2}{N_1}}s^{-2}ds\|h\|_X^3\\ \lesssim&\|h\|_X^3,\end{aligned}$$

where we need to take $N_1$ sufficiently large.

We turn now to the estimate of (6.12). We need a lemma

**Lemma 6.2.** *We have*

$$\|e^{it\langle\nabla\rangle}\partial_t f(t)\|_{\infty-}\lesssim\langle t\rangle^{-2+C\epsilon_1}\|h\|_X^2.$$

*Proof of Lemma 6.2.* By the derivations in the introduction section (see (1.10)–(3.4)), we can write

$$e^{it\langle\nabla\rangle}\partial_t f=\mathcal{R}T_{O(\langle\xi\rangle)}(\mathcal{R}h,\mathcal{R}h).$$

Hence

$$\begin{aligned}\|e^{it\langle\nabla\rangle}\partial_t f\|_{\infty-}&\lesssim\|\langle\nabla\rangle^{N_1}h(t)\|_{\infty-}^2\\ &\lesssim\langle t\rangle^{-2+C\epsilon_1}\|h\|_X^2.\end{aligned}$$

□

Now we continue to estimate (6.12). By Lemma 6.2,

$$\begin{aligned}&\|\mathcal{F}^{-1}((6.12))\|_{2-\epsilon_1}\\ \lesssim&\int_1^t s^{1+C\epsilon_1+c\delta_1+C\delta_2}s^{-3}ds\|h\|_X^4\\ \lesssim&\|h\|_X^4.\end{aligned}$$

Similarly

$$\|\mathcal{F}^{-1}((6.13))\|_{2-\epsilon_1}\lesssim\|h\|_X^4.$$

We have concluded the estimate of Subcase 3b.

`Subcase 3c`: $|\eta|\le\langle s\rangle^{-\delta_2}$, $|\xi|>\langle s\rangle^{-\delta_2/N_1}$ and $|\sigma|\ge\langle s\rangle^{-\delta_2/10}$.

Then clearly

$$|2\sigma-\eta|\gtrsim\langle s\rangle^{-\delta_2/10}.$$

By (6.6), we then have

$$|\partial_\sigma \phi| = \left| \frac{\sigma - \eta}{\langle \sigma - \eta \rangle} + \frac{\sigma}{\langle \sigma \rangle} \right| \gtrsim \langle s \rangle^{-\delta_2}.$$

Hence we can do integration by parts in $\sigma$ in (5.9), i.e.,

$$-ise^{-is\phi} = \frac{\partial_\sigma \phi}{|\partial_\sigma \phi|^2} \cdot \partial_\sigma (e^{-is\phi}).$$

This will produce some additional Fourier symbols. However they are all separable in the sense of (5.3). This yields

$$\begin{aligned} \left\| \mathcal{F}^{-1}((5.9)) \right\|_{2-\epsilon_1} &\lesssim \int_1^t \langle s \rangle^{C(\delta_1+\epsilon_1+\delta_2)-2} ds \|h\|_X^3 \\ &\lesssim \|h\|_X^3. \end{aligned}$$

We now consider

`Subcase 3d`: $|\eta| \geq \langle s \rangle^{-\delta_2}$. In this case we have to exploit some delicate cancelations of the phases.

By Lemma 2.4, we can write

$$\partial_\xi \phi = Q_1(\xi, \eta, \sigma) \partial_\eta \phi + Q_2(\xi, \eta, \sigma) \partial_\sigma \phi, \tag{6.14}$$

where $Q_1$, $Q_2$ are smooth symbols. The idea then is to do integration by parts in $\eta$ or $\sigma$. To simplify matters, we only present the case of integration by parts in $\eta$ since the other case will be similar. We stress that one has to use the localization $|\eta| \gtrsim \langle s \rangle^{-\delta_2}$ because the Riesz symbol $\eta/|\eta|$ will then be smooth.

$$\begin{aligned} \|\mathcal{F}^{-1}((5.9))\|_{2-\epsilon_1} \lesssim & \int_1^t \langle s \rangle^{C\delta_1+C\delta_2+C\epsilon_1} \|P_{\lesssim \langle s \rangle^{\delta_1}} e^{is\langle \nabla \rangle} (xf)\|_{2+\epsilon_1} \|h\|_{\infty-}^2 ds \\ & + \int_1^t \langle s \rangle^{C\delta_1+C\delta_2+C\epsilon_1} \|P_{\lesssim \langle s \rangle^{\delta_1}} e^{is\langle \nabla \rangle} |\nabla|^{-1} f\|_{2+100\epsilon_1} \|h\|_{\infty-}^2 ds \\ & + \int_1^t \langle s \rangle^{C\delta_1+C\delta_2+C\epsilon_1} \|h\|_2 \|h\|_{\infty-}^2 ds \\ \lesssim & \int_1^t \langle s \rangle^{C\epsilon_1+C\delta_1+C\delta_2} \langle s \rangle^{-2} ds \|h\|_X^3 \\ \lesssim & \|h\|_X^3. \end{aligned}$$

We have finished all estimates corresponding to the phase (6.6).

`Case 4`:

$$\phi(\xi, \eta, \sigma) = \langle \xi \rangle + \langle \xi - \eta \rangle + \langle \eta - \sigma \rangle - \langle \sigma \rangle. \tag{6.15}$$

Although in this case we note that

$$|\phi| \gtrsim 1,$$

it is not good to do integrate by parts in the time variable $s$ in the whole phase since the Riesz symbol $\eta/|\eta|$ is not necessarily smooth near the origin.

We discuss two subcases.

`Subcase 4a`: $|\eta| \leq \langle s \rangle^{-\delta_2}$. In this regime it is not good to integrate by part the whole phase due to the Riesz symbol $\eta/|\eta|$. To resolve this difficulty, note that

$$\langle \eta - \sigma \rangle - \langle \sigma \rangle = Q(\eta, \sigma)\eta,$$

where $Q(\eta, \sigma)$ is smooth. We then use the identity

$$\frac{1}{i(\langle\xi\rangle + \langle\xi-\eta\rangle)} \frac{d}{ds}\Big(e^{is(\langle\xi\rangle+\langle\xi-\eta\rangle)}\Big) \\ = e^{is(\langle\xi\rangle+\langle\xi-\eta\rangle)}$$

to do integration by parts in the time variable $s$. We will have no trouble when the time derivative hits the term $e^{is(\langle\eta-\sigma\rangle-\langle\sigma\rangle)}$ since

$$\frac{d}{ds}\Big(e^{is(\langle\eta-\sigma\rangle-\langle\sigma\rangle)}\Big) \\ = Q(\eta,\sigma)\eta e^{is(\langle\eta-\sigma\rangle-\langle\sigma\rangle)}.$$

The $\eta$ factor will produce an extra decay of $s^{-\delta_2}$ by using $\partial P_{\lesssim\langle s\rangle^{-\delta_2}} \approx s^{-\delta_2}$ when we perform $L_p$ estimates. When the time derivative hits $\hat{f}(s)$ we will get an additional decay of $s^{-1+}$ which is certainly enough for us. Note that all the symbols are separable in the sense of (5.3). The details are quite similar (in fact simpler) to that of Case 3. We therefore skip it here.

`Subcase 4b:` $|\eta| \gtrsim \langle s\rangle^{-\delta_2}$. In this case we can directly integrate by parts in the time variable $s$ since the Riesz symbol $\eta/|\eta|$ is smooth in this regime. The calculations are quite straightforward using Lemma 6.2. There are also boundary terms when we perform integration by parts in $s$ but they can be easily estimated along similar lines as done in Case 3. Hence we skip it.

This concludes the estimate of phase (6.15).

`Case 5:`

$$\phi(\xi,\eta,\sigma) = \langle\xi\rangle + \langle\xi-\eta\rangle - \langle\eta-\sigma\rangle + \langle\sigma\rangle.$$

This is almost the same as Case 4 after the change of variable $\sigma \to \eta - \sigma$. Hence we skip it.

`Case 6:`

$$\phi(\xi,\eta,\sigma) = \langle\xi\rangle - \langle\xi-\eta\rangle + \langle\eta-\sigma\rangle + \langle\sigma\rangle. \tag{6.16}$$

Again we sketch the proof and discuss two subcases.

`Subcase 6a:` $|\eta| \le \langle s\rangle^{-\delta_2}$. Observe that

$$\partial_\xi\phi = Q(\xi,\eta)\cdot\eta$$

where $Q$ is smooth. The extra $\eta$ factor allows us to gain one derivative and $s^{-\delta_2}$ decay. This is similar to Case 3a and hence we omit the details.

`Subcase 6b:` $|\eta| > \langle s\rangle^{-\delta_2}$. In this regime the Riesz symbol $\eta/|\eta|$ is smooth and hence we can perform integration by parts in the time variable $s$. The estimates are quite similar to Case 4b. We omit the details.

`Case 7:`

$$\phi(\xi,\eta,\sigma) = \langle\xi\rangle + \langle\xi-\eta\rangle + \langle\eta-\sigma\rangle + \langle\sigma\rangle.$$

`Subcase 7a:` $|\eta| \le \langle s\rangle^{-\delta_2}$ and $|\sigma| \le \langle s\rangle^{-\delta_2/10}$. In this case we again will do integration by parts using *part of the phase*. Write

$$e^{-is\phi} = \frac{-i}{\langle\xi\rangle + \langle\xi-\eta\rangle + 2}\partial_s\Big(e^{-is(\langle\xi\rangle+\langle\xi-\eta\rangle+2)}\Big)e^{-is(\langle\eta-\sigma\rangle+\langle\sigma\rangle-2)}.$$

Note that

$$\langle\eta-\sigma\rangle + \langle\sigma\rangle - 2 = b_4(\eta,\sigma)\cdot(\eta-\sigma) + b_5(\eta,\sigma)\cdot\sigma,$$

where $b_4$ and $b_5$ are smooth. The estimates are similar to before and hence we omit the details.

`Subcase 7b:` $|\eta| \leq \langle s\rangle^{-\delta_2}$ and $|\sigma| > \langle s\rangle^{-\delta_2/10}$. In this case

$$|\partial_\sigma \phi| \gtrsim \langle s\rangle^{-\delta_2/10}$$

and we can integrate by parts in $\sigma$. We omit details.

`Subcase 7c:` $|\eta| > \langle s\rangle^{-\delta_2}$. In this case the Riesz symbol $\eta/|\eta|$ is smooth and we can directly integrate by parts in the whole phase $\phi$. The details are omitted.

We have finished the estimates of all cases. The proposition is now proved. □

## 7. Proof of Theorem 1.2

In this section we complete the proof of Theorem 1.2.

*Proof of Theorem 1.2.* We begin with the $H^N$-energy estimate. By (1.10) and energy estimates, one obtains

$$\frac{d}{dt}(\|h(t)\|_{H^N}^2) \lesssim (\|\nabla h(t)\|_\infty + \|\langle\nabla\rangle u(t)\|_\infty + \|\partial \mathbf{v}(t)\|_\infty)\|h(t)\|_{H^N}^2.$$

By splitting into low and high frequencies, we then have

$$\begin{aligned}
&\|\nabla h(t)\|_\infty + \|\langle\nabla\rangle u(t)\|_\infty + \|\partial \mathbf{v}(t)\|_\infty\\
\lesssim&\|\nabla h(t)\|_\infty + \||\nabla| h(t)\|_\infty + \||\nabla|^{-1}\partial_i\partial_j h\|_\infty\\
\lesssim&\||\nabla|^{\frac12}\langle\nabla\rangle h(t)\|_\infty.
\end{aligned}$$

Then clearly as long as $L^\infty$ norm of $|\nabla|^{\frac12}\langle\nabla\rangle h(t)$ decay at the sharp rate $\frac{1}{\langle t\rangle}$, we can close the energy estimate and obtain $\|h(t)\|_{H^N} \lesssim t^{\delta_1}$. We stress here that it is very important to have some amount of derivatives on $h$ especially in the low frequency regime due to the non-locality caused by the Riesz transform.

We then only need to estimate $\|h(t)\|_{X_1}$ (see (1.12)). For this we need to use the fine decomposition (3.9) and estimate $g$ and $f_{\text{cubic}}$ separately. The estimate of $g$ was given in Section 4. The cubic interaction $f_{\text{cubic}}$ was estimated in Section 5 and 6. To simplify the presentation we have suppressed many unnecessary details (and repetitive computations). Finally we note that scattering in the intermediate energy space $H^{N'}$ follows from calculations in Section 4–6 and hence we will omit the proof. The theorem is proved. □

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Department of Mathematics, University of Iowa, 14 MacLean Hall, Iowa City, IA 52242

*E-mail address*: mpdongli@gmail.com

Institute of Applied Physics and Computational Mathematics, P. O. Box 8009, Beijing, China, 100088,

*E-mail address*: yerfmath@yahoo.cn